1.6.3 SPECTRAL ANALYSIS VIA THE DFT

An important application of the DFT is to numerically determine the spectral content of signals.

However, the extent to which this is possible is limited by two factors:

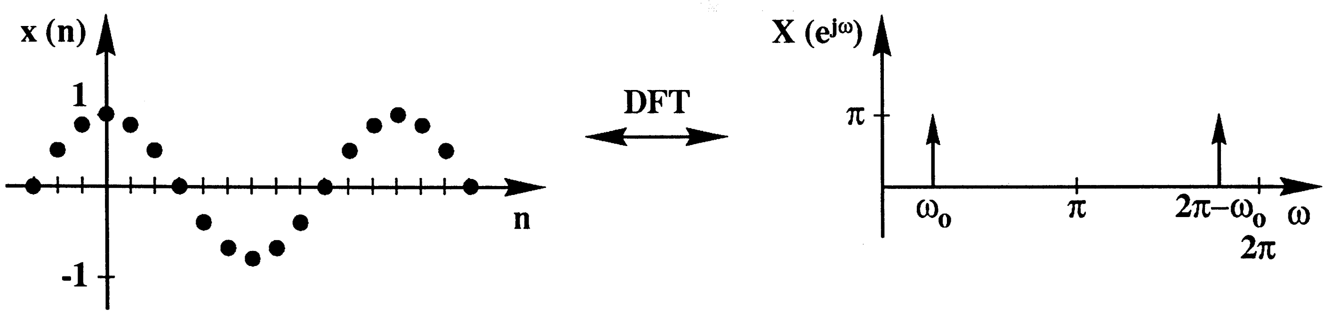
1. truncation of the signal - causes *leakage*
2. frequency domain sampling - causes *picket fence effect*

Truncation of the Signal

Let $x(n) = \cos(\omega_0 n)$

Recall that

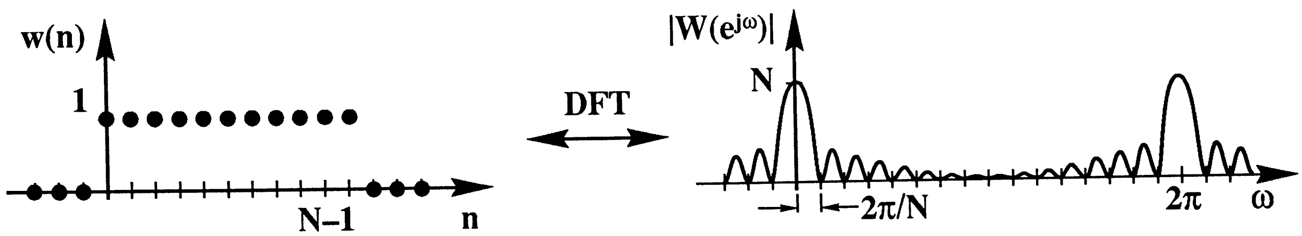
$$X(e^{j\omega}) = \text{rep}_{2\pi}[\pi\delta(\omega - \omega_0) + \pi\, \delta(\omega + \omega_0)]$$



Also, let $$w(n) = \begin{cases} 1\,, & 0 \le n \le N-1 \\ 0\,, & \text{else} \end{cases}$$

Recall $$W(e^{j\omega}) = e^{-j\omega(N-1)/2} \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

Note that $W(e^{j\omega}) = W(e^{j(\omega + 2\pi m)})$ for all integer m.



Now let $x_t(n) = x(n)\, w(n)$ t - truncated

By the product theorem

$$X_t(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} W(e^{j(\omega - \mu)})\, X(e^{j\mu})\, d\mu$$

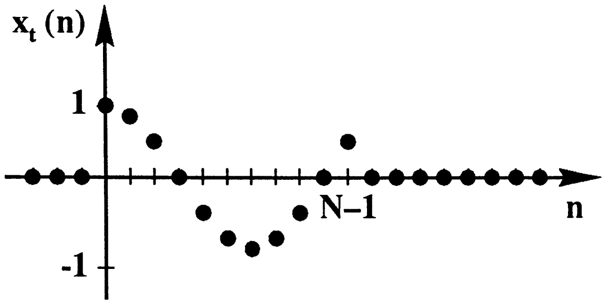
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} W(e^{j(\omega - \mu)})\, \mathrm{rep}_{2\pi}[\pi\delta(\mu - \omega_0) + \pi\delta(\mu + \omega_0)]\, d\mu$$

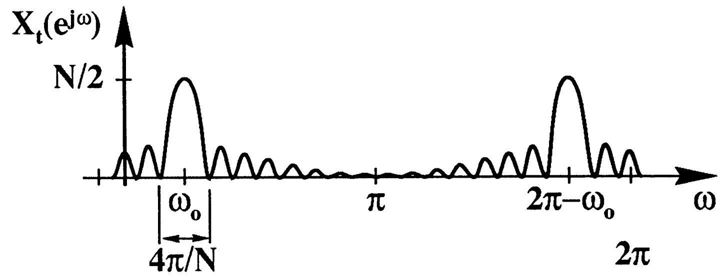
$$X_t(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} W(e^{j(\omega - \mu)})[\pi\delta(\mu - \omega_0) + \pi\delta(\mu + \omega_0)]\, d\mu$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} W(e^{j(\omega - \mu)})\, \delta(\mu - \omega_0)\, d\mu$$

$$+ \int_{-\pi}^{\pi} W(e^{j(\omega - \mu)})\, \delta(\mu + \omega_0)\, d\mu$$

$$X_t(e^{j\omega}) = \frac{1}{2}\left[W(e^{j(\omega-\omega_0)}) + W(e^{j(\omega+\omega_0)})\right]$$





Frequency Domain Sampling

Let

$$x_r(n) = \sum_m x_t(n + mN) \qquad \text{r—replicated}$$

Recall from relation between DFT and DTFT of finite length sequence that

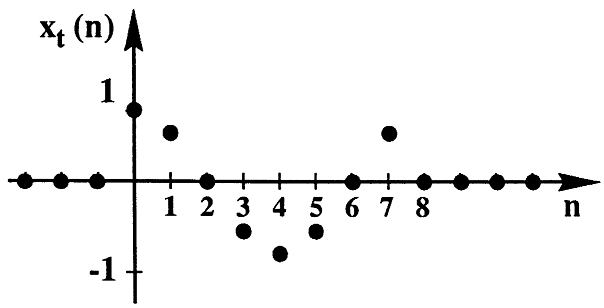
$$\underset{\text{DFT}}{X_r(k)} = \underset{\text{DTFT}}{X_t(e^{j2\pi k/N})}$$

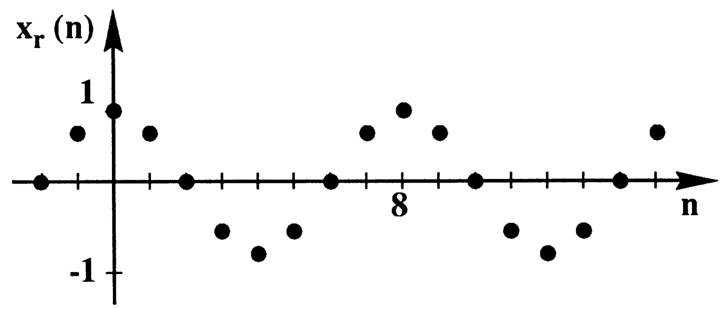
$$= \frac{1}{2}\left[W(e^{j(2\pi k/N - \omega_0)}) + W(e^{j(2\pi k/N + \omega_0)})\right]$$

Consider two cases:

1. $\omega_0 = 2\pi k_0/N$ for some integer k_0

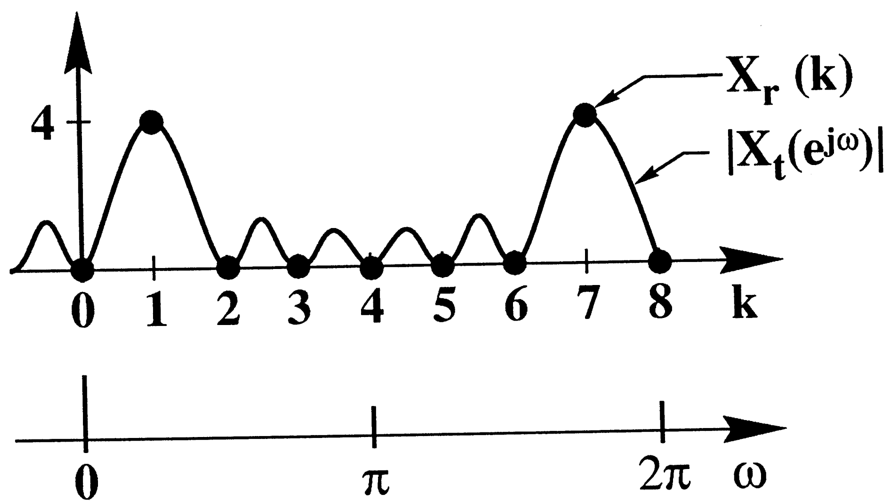
   Example: $k_0 = 1$, $N = 8$, $\omega_0 = \pi/4$





$$X_r(k) = \frac{1}{2\pi}[W(e^{j2\pi(k-k_0)/N}) + W(e^{j2\pi(k+k_0)/N})]$$

Example: $k_0 = 1, \quad N = 8$



In this case,

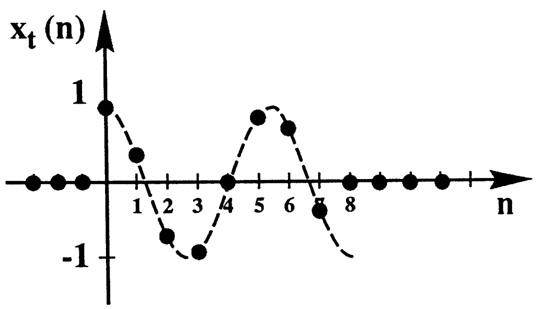
$$X_r(k) = \frac{N}{2}\,[\delta(k - k_0) + \delta(k - (N - k_0))]\,,\quad 0 \le k \le N-1$$

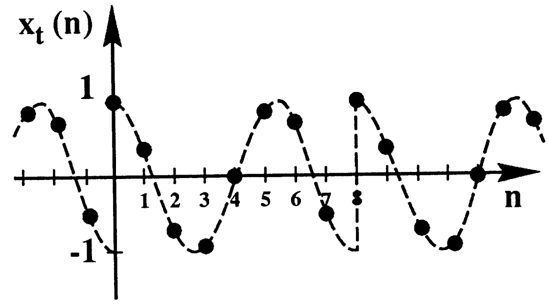
as shown before.

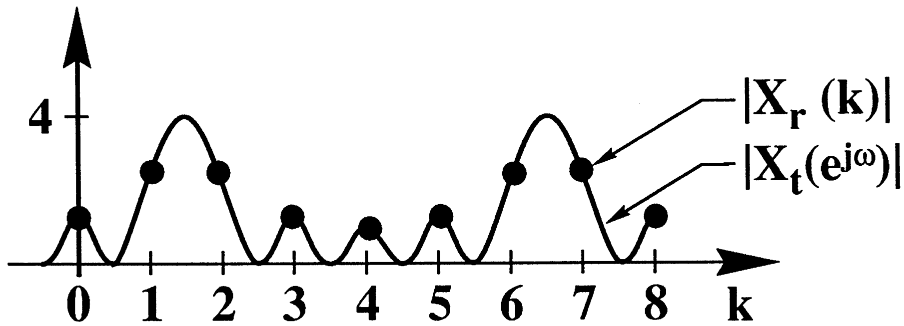
There is no picket fence effect.

2. $\omega_0 = 2\pi k_0/N + \pi/N$ for some integer k_0

Example: $k_0 = 1, \quad N = 8, \quad \omega_0 = 3\pi/8$







picket fence effect:

— spectral peak is midway between sample locations

— each sidelobe peak occurs at a sample location

- Both truncation and picket fence effects are reduced by increasing N.

- Sidelobes may be suppressed at the expense of a wider mainlobe by using a smoothly tapering window.