APPENDIX C

EXPECTED VALUE OF PRODUCTS OF MOMENTS

To minimize the algebra, this derivation is done in cases. Some preliminary results are necessary, however. From Appendix A,

$$M_k M_\ell = (j2\pi)^{-(\ell+k)}(T)^{-2} \int_t \int_\tau \int_u \int_v \xi(t)\xi^*(\tau)\xi(u)\xi^*(v)$$
$$\cdot\ G_T(t)G_T(\tau)G_T(u)G_T(v)$$
$$\cdot\ \delta^{(k)}(\tau - t)\delta^{(\ell)}(v - u)dvdud\tau dt. \qquad (C.1)$$

Therefore,

$$E[M_k M_\ell] = (j2\pi)^{-(\ell+k)}(T)^{-2}$$
$$\cdot \int_t \int_\tau \int_u \int_v E[\xi(t)\xi^*(\tau)\xi(u)\xi^*(v)]G_T(t)G_T(\tau)$$
$$\cdot\ G_T(u)G_T(v)\delta^{(k)}(\tau - t)\delta^{(\ell)}(v - u)dvdud\tau dt.$$
$$(C.2)$$

But, for a gaussian process

$$E[x_1x_2x_3x_4] = E[x_1x_2]E[x_3x_4]$$
$$+ E[x_1x_3]E[x_2x_4] + E[x_1x_4]E[x_2x_4]$$

and from the results of Swartz, Bennet, and Stein (11)

$$E[\xi(t)\xi(\tau)] = 0.$$

So, if z(t) is a gaussian process, then

$$E[\xi(t)\xi^*(\tau)\xi(u)\xi^*(v)] = R_\xi(t - \tau)R_\xi(u - v)$$

$$+ R_\xi(t - v)R_\xi(u - \tau). \quad (C.3)$$

Substitution of this result into Equation (C.2) and a change of the order of integration yields

$$E[M_k M_\ell] = (j2\pi)^{-(\ell+k)}T^{-2}$$

$$\cdot \int_t \int_u \int_\tau \int_v [R_\xi(t - \tau)R_\xi(u - v)$$

$$+ R_\xi(t - v)R_\xi(u - \tau)]G_T(t)G_T(u)$$

$$\cdot G_T(\tau)G_T(v)\delta^{(k)}(\tau - t)\delta^{(\ell)}(v - u)dvd\tau dudt.$$

$$(C.4)$$

<u>Case 1: k = 0, ℓ = 0</u>

$$E[M_0^2] = T^{-2} \int_t \int_u G_T(t)G_T(u)[|R_\xi(0)|^2 + |R_\xi(t - u)|^2 dudt$$

$$E[M_0^2] = P^2 + T^{-2} \int_{t = -\frac{T}{2}}^{\frac{T}{2}} \int_{u = -\frac{T}{2}}^{\frac{T}{2}} |R_\xi(t - u)|^2 dudt. \quad (C.5)$$

Writing $R_\xi(\tau)$ in terms of $S_\xi(f)$ yields

$$E[M_0^2] = P^2 + T^{-2} \int_{t = -\frac{T}{2}}^{\frac{T}{2}} \int_{u = -\frac{T}{2}}^{\frac{T}{2}} \int_f \int_g S_\xi(f)S_\xi(g)$$

$$\cdot e^{j2\pi f(t-u)}e^{j2\pi g(u-t)}dgdfdudt. \quad (C.6)$$

Interchanging the order of integration yields

$$E[M_0^2] = P^2 + T^{-2}$$
$$\cdot \int_f \int_g S_\xi(f) S_\xi(g) \left[\int_{t=-\frac{T}{2}}^{\frac{T}{2}} e^{j2\pi(f-g)t} dt \right]$$
$$\cdot \left[\int_{u=-\frac{T}{2}}^{\frac{T}{2}} e^{j2\pi(g-f)u} du \right] dg df. \qquad (C.7)$$

Integrating over t and u yields

$$E[M_0^2] = P^2 + T^{-1} \int_f \int_g S_\xi(f) S_\xi(g) \frac{\sin^2 2\pi(g-f)\frac{T}{2}}{[2\pi(g-f)]^2 T} dg df. \qquad (C.8)$$

For large T

$$\frac{\sin^2 2\pi(g-f)\frac{T}{2}}{[2\pi(g-f)]^2 T} \rightarrow \delta(g-f).$$

Therefore, for large T,

$$E[M_0^2] = P^2 + T^{-1} \int_f S_\xi^2(f) df. \qquad (C.9)$$

Case 2: $k = 0, \ell = 1$

$$E[M_0 M_1] = (j2\pi)^{-1} T^{-2}$$
$$\cdot \int_t \int_u G_T(t) G_T(u) \{ [R_\xi(0) \dot{R}_\xi^*(0) + R_\xi(t-u) \dot{R}^*(t-u)] \qquad (C.10)$$
$$+ [|R_\xi(0)|^2 + |R_\xi(t-u)|^2] G_T(t) \dot{G}_T(u) \} du dt.$$

Using methods similar to those in the derivation of $E[M_0^2]$, the first integral can be evaluated. Therefore,

$$E[M_0M_1] = f_aP^2 + T^{-1}$$

$$\cdot \int_f \int_g S_\xi(f)S_\xi(g) \frac{\sin^2\left[2\pi(f - g)\frac{T}{2}\right]}{[2\pi(f - g)]^2T} \, dgdf$$

$$+ (j2\pi)^{-1}T^{-2} \int_t \int_u G_T(t)G_T(u)\dot{G}_T(u)$$

$$\cdot [P^2 + |R_\xi(t - u)|^2]dudt. \qquad (C.11)$$

The contribution to the second integral from the P^2 term is zero by a symmetry argument. Again, writing $R_\xi(\tau)$ in terms of $S_\xi(f)$

$$E[M_0M_1] = f_aP^2 + T^{-1}$$

$$\cdot \int_f \int_g gS_\xi(f)S_\xi(g) \frac{\sin^2\left[2\pi(f - g)\frac{T}{2}\right]}{[2\pi(f - g)]^2T} \, dgdf$$

$$+ (j2\pi)^{-1}T^{-2} \int_f \int_g \int_t \int_u G_T(t)G_T(u)\dot{G}_T(u)$$

$$\cdot S_\xi(f)S_\xi(g)e^{j2\pi(f-g)(t-u)}dudtdgdf. \qquad (C.12)$$

Integrating with respect to u and t yields

$$E[M_0M_1] = f_aP^2 + T^{-1}$$

$$\cdot \int_f \int_g gS_\xi(f)S_\xi(g) \frac{\sin^2\left[2\pi(f - g)\frac{T}{2}\right]}{[2\pi(f - g)]^2T} \, dgdf$$

$$+ 2\pi T^{-1} \int_f \int_g (f - g)^2S_\xi(f)S_\xi(g)$$

$$\cdot \frac{\sin^2\left[2\pi(f - g)\frac{T}{2}\right]}{[2\pi(f - g)]^2 T} \, dgdf. \qquad (C.13)$$

Therefore, for large T, the second integral is zero and

$$E[M_0 M_1] = f_a P^2 + T^{-1} \int_f f S_\xi^2(f) df. \qquad (C.14)$$

<u>Case 3: $k = \ell = 1$</u>

From (C.4)

$$E[M_1^2] = (j2\pi)^{-2} T^{-2} \int_t \int_u G_T(t) G_T(u) \{[\dot{R}_\xi(0)\dot{R}_\xi^*(0) + \dot{R}_\xi(t - u)\dot{R}_\xi^*(t - u)] + \dot{G}_T(t)\dot{G}_T(u)[|R_\xi(0)|^2 + |R_\xi(t - u)|^2]\} dudt. \qquad (C.15)$$

The derivation is, again, similar and, after some algebra

$$E[M_1^2] = (f_a P)^2 + T^{-1} \cdot \int_f \int_g fg S_\xi(f) S_\xi(g) \frac{\sin^2\left[2\pi(f - g)\frac{T}{2}\right]}{[2\pi(f - g)]^2 T} \, dgdf + 4\pi^2 \int_f \int_g (f - g)^4 S_\xi(f) S_\xi(g) \cdot \frac{\sin^2\left[2\pi(f - g)\frac{T}{2}\right]}{[2\pi(f - g)^2] T} \, dgdf \;. \qquad (C.16)$$

Again, for large T the second integral is zero and

$$E[M_1^2] = (f_a P)^2 + T^{-1} \int_f f^2 S_\xi^2(f) df. \qquad (C.17)$$

Some results are necessary for the work of the next two cases. From Appendices A, B

$$E[M_k(M_2 - \hat{P}B_T^2)] = (j2\pi)^{-(k+2)}T^{-2}$$

$$\cdot \left\{ \int_t \int_\tau \int_u \int_v G_T(t)G_T(\tau)G_T(u)G_T(v) \right.$$

$$\cdot\ \delta^{(k)}(\tau - t)\ddot{\delta}(v - u)$$

$$\cdot\ E[\xi(t)\xi^*(\tau)\xi(u)\xi^*(v)]dvdud\tau dt$$

$$- \int_t \int_\tau \int_u G_T(t)G_T(\tau)G_T(u)\delta^{(k)}(\tau - t)$$

$$\left. \cdot\ E[\xi(t)\xi^*(t)\xi(u)\xi^*(u)]\ddot{G}_T(u)dud\tau dt \right\}. \tag{C.18}$$

Assuming a gaussian process, integrating with respect to v in the first integral, and combining terms yields

$$E[M_k(M_2 - \hat{P}B_T^2)] = (j2\pi)^{-(k+2)}T^{-2}$$

$$\cdot \int_t \int_\tau \int_u G_T(t)G_T(\tau)G_T(u)\delta^{(k)}(\tau - t)$$

$$\cdot\{G_T(u)[R_\xi(t - \tau)\ddot{R}_\xi^*(0)$$

$$+ \dot{R}_\xi(t - u)R_\xi^*(\tau - u)]$$

$$+ [\ddot{G}(u) - \ddot{G}(u)][R_\xi(t - \tau)R_\xi^*(0)$$

$$+ R_\xi(t - u)R_\xi^*(\tau - u)]\}dudtd\tau. \tag{C.19}$$

Or, interchanging the order of integration

$$E[M_k(M_2 - \hat{P}B_T^2)] = (j2\pi)^{-(k+2)}T^{-2}$$

$$\cdot \int_t \int_u \int_\tau G_T(t)G_T(\tau)G_T(u)\delta^{(k)}(\tau - t)$$

$$\cdot \{G_T(u)[R_\xi(t - \tau)\ddot{R}_\xi^*(0)$$

$$+ \ddot{R}_\xi(t - u)R_\xi^*(\tau - u)]$$

$$+ 2\dot{G}_T(u)[R_\xi(t - \tau)\dot{R}_\xi^*(0)$$

$$+ \dot{R}_\xi(t - u)R_\xi^*(\tau - u)]\}d\tau du dt. \tag{C.20}$$

Case 4: $k = 0, \ell = 2$

Performing the integration over τ yields

$$E[M_0(M_2 - \hat{P}B_T^2)] = (j2\pi)^{-2}T^{-2}$$

$$\cdot \int_t \int_u G_T(t)G_T(u)\{R_\xi(0)\ddot{R}_\xi^*(0)$$

$$+ \ddot{R}_\xi(t - u)R_\xi^*(t - u) + 2\dot{G}_T(u)[R_\xi(0)R_\xi^*(0)$$

$$+ \dot{R}_\xi(t - u)R_\xi^*(t - u)]\}du dt. \tag{C.21}$$

Again, switching to the frequency domain and integrating with respect to t and u yields

$$E[M_0(M_2 - \hat{P}B_T^2)] = P^2[B^2 + B_\rho^2 + f_a^2]$$

$$+ T^{-1}\left\{\int_f \int_g g^2 S_\xi(f)S_\xi(g)\right.$$

$$\cdot \frac{\sin^2\left[2\pi(f - g)\frac{T}{2}\right]}{[2\pi(f - g)]^2 T} \, dgdf$$

$$+ 2\pi \int_f \int_g f(f - g) S_\xi(f) S_\xi(g)$$

$$\left. \cdot \frac{\sin^2\left[2\pi(f - g)\frac{T}{2}\right]}{[2\pi(f - g)]^2 T} \, dgdf \right\} . \quad (C.22)$$

For large T, the second integral is zero and

$$E[M_0(M_1 - \hat{P}B_T^2)] = P^2[B^2 + B_\rho^2 + f_a^2] + T^{-1} \int_f f^2 S_\xi^2(f) df. \quad (C.23)$$

Case 5: $k = 1, \ell = 2$

From Equation (C.20)

$$E[M_1(M_2 - \hat{P}B_T^2)] = (j2\pi)^{-3} T^{-2}$$

$$\cdot \int_t \int_u G_T(t) G_T(u)$$

$$\cdot \{\dot{G}_T(t)[\dot{R}_\xi(0)\ddot{R}_\xi^*(0)$$

$$+ \ddot{R}_\xi(t - u)\dot{R}_\xi^*(t - u) + 2\dot{G}_T(u)\dot{R}_\xi(0)\dot{R}_\xi^*(0)$$

$$+ 2\dot{G}_T(u)\dot{R}_\xi(t - u)\dot{R}_\xi^*(t - u)]$$

$$+ G_T(t)[\dot{R}_\xi(0)\ddot{R}_\xi^*(0) + \ddot{R}_\xi(t - u)\dot{R}_\xi^*(t - u)$$

$$+ 2\dot{G}_T(u)\dot{R}_\xi(0)\dot{R}_\xi^*(0)$$

$$+ 2\dot{G}_T(u)\dot{R}_\xi(t - u)\dot{R}_\xi^*(t - u)]\} \, dudt. \quad (C.24)$$

The four terms having zero for an argument of the autocorrelation function can be integrated giving zero for three of the terms and leaving

$$E[M_1(M_2 - \hat{P}B_T^2)] = f_a P^2[B^2 + B_\rho^2 + f_a^2] + (j2\pi)^{-3}T^{-2}$$

$$\cdot \int_t \int_u G_T(t)G_T(u)$$

$$\cdot \{\dot{G}_T(u)\ddot{R}_\xi(t - u)\dot{R}_\xi^*(t - u)$$

$$+ 2\dot{G}_T(t)\dot{G}_T(u)\dot{R}_\xi(t - u)\dot{R}_\xi^*(t - u)$$

$$+ \ddot{R}_\xi(t - u)\dot{R}_\xi^*(t - u)$$

$$+ 2\dot{G}_T(u)\dot{R}_\xi(t - u)\dot{R}_\xi^*(t - u)\}dudt. \tag{C.25}$$

Switching to the frequency domain and interchanging the order of integration yields

$$E[M_1(M_2 - \hat{P}B_T^2)] = f_a P^2[B^2 + B_\rho^2 + f_a^2] + (j2\pi)^{-3}T^{-2}$$

$$\cdot \int_f \int_g S_\xi(f)S_\xi(g) \int_t \int_u G_T(t)G_T(u)$$

$$\cdot \{\dot{G}_T(t)(j2\pi f)^2(j2\pi g)$$

$$+ 2\dot{G}_T(t)\dot{G}_T(u)(j2\pi f)(j2\pi g) \tag{C.26}$$

$$+ (j2\pi f)^2(j2\pi g) + 2\dot{G}_T(u)$$

$$\cdot (j2\pi f)(j2\pi g)\}e^{j2\pi(f-g)(t-u)}dudtdgdf.$$

Again, when T is large any term with $\dot{G}_T(x)$ in it is zero and

$$E[M_1(M_2 - \hat{P}B_T^2)] = f_a P^2[B^2 + B_\rho^2 + f_a^2] + T^{-1} \int_f f^3 S_\xi^2(f)df. \tag{C.27}$$

Case 6: $k = 2, \ell = 2$

From Appendix B,

$$\begin{aligned} E[(M_2 - \hat{P}B_T^2)^2] = (2\pi)^{-4}T^{-2} \int_t \int_u G_T(t)G_T(u) \\ \cdot \Big\{ \int_\tau \int_v E[\xi(t)\xi^*(\tau)\xi(u)\xi^*(v)] \\ \cdot G_T(\tau)G_T(v)\ddot{\delta}(\tau - t)\ddot{\delta}(v - u)dvd\tau \\ - \int_\tau E[\xi(t)\xi^*(\tau)\xi(u)\xi^*(u)] \\ \cdot G_T(\tau)\ddot{G}_T(u)\ddot{\delta}(\tau - t)d\tau \\ - \int_v E[\xi(u)\xi^*(v)\xi(t)\xi^*(t)] \\ \cdot G_T(v)\ddot{G}_T(t)\ddot{\delta}(v - u)dv \\ + E[\xi(t)\xi^*(t)\xi(u)\xi^*(u)]\ddot{G}_T(t)\ddot{G}_T(u)\Big\} dudt. \end{aligned} \tag{C.28}$$

Assuming a gaussian process and integrating with respect to τ and v yields

$$E[(M_2 - \hat{P}B_T^2)^2] = (2\pi)^{-4}T^{-2} \int_t \int_u G_T(t)G_T(u)$$

$$\cdot \{[R_\xi(0)R_\xi^*(0) + R(t - u)R_\xi^*(t - u)]$$

$$\cdot \ddot{G}_T(t)\ddot{G}_T(u)$$

$$+ [\dot{R}_\xi(0)R_\xi^*(0) + R(t - u)\dot{R}_\xi^*(t - u)]$$

$$\cdot \dot{G}_T(t)\ddot{G}_T(u)$$

$$+ [\ddot{R}_\xi(0)R_\xi^*(0) + R(t - u)\ddot{R}_\xi^*(t - u)]$$

$$\cdot G_T(t)\ddot{G}_T(u)$$

$$+ [R_\xi(0)\dot{R}_\xi^*(0) + R(t - u)R_\xi^*(t - u)]$$

$$\cdot \ddot{G}_T(t)\dot{G}_T(u)$$

$$+ [\dot{R}_\xi(0)\dot{R}_\xi^*(0) + \dot{R}_\xi(t - u)R_\xi^*(t - u)]$$

$$\cdot \dot{G}_T(t)\dot{G}_T(u)$$

$$+ [\ddot{R}_\xi(0)\dot{R}_\xi^*(0) + \dot{R}_\xi(t - u)\ddot{R}_\xi^*(t - u)]$$

$$\cdot G_T(t)\dot{G}_T(u)$$

$$+ [R_\xi(0)\ddot{R}_\xi^*(0) + \ddot{R}_\xi^*(t - u)R_\xi^*(t - u)]$$

$$\cdot \ddot{G}_T(t)G_T(u)$$

$$+ [\dot{R}_\xi(0)\ddot{R}^*_\xi(0) + \ddot{R}_1(t - u)\dot{R}^*_\xi(t - u)]$$

$$\cdot \dot{G}_T(t)G_T(u)$$

$$+ [\ddot{R}_\xi(0)\ddot{R}^*_\xi(0) + \ddot{R}_\xi(t - u)\ddot{R}^*_\xi(t - u)]$$

$$\cdot G_T(t)G_T(u)$$

$$- [R_\xi(0)R^*_\xi(0) + R(t - u)R^*_\xi(t - u)]$$

$$\cdot \ddot{G}_T(t)\ddot{G}_T(u)$$

$$- [\dot{R}_\xi(0)R^*_\xi(0) + R_\xi(t - u)\dot{R}^*_\xi(t - u)]$$

$$\cdot \dot{G}_T(t)\ddot{G}_T(u)$$

$$- [\ddot{R}_\xi(0)R^*_\xi(0) + \ddot{R}_\xi(t - u)R^*_\xi(t - u)]$$

$$\cdot G_T(t)\ddot{G}_T(u)$$

$$- [R_\xi(0)R^*_\xi(0) + R_\xi(t - u)R^*_\xi(t - u)]$$

$$\cdot \ddot{G}_T(t)\ddot{G}_T(u)$$

$$- [R_\xi(0)\dot{R}^*_\xi(0) + \dot{R}_\xi(t - u)R^*_\xi(t - u)]$$

$$\cdot \ddot{G}_T(t)\dot{G}_T(u)$$

$$- [R_\xi(0)\ddot{R}^*_\xi(0) + \ddot{R}_\xi(t - u)R^*_\xi(t - u)]$$

$$\cdot \ \ddot{G}_T(t)G_T(u)$$

$$+ \ [R_\xi(0)R_\xi^*(0) + R_\xi(t - u)R_\xi^*(t - u)]$$

$$\cdot \ \ddot{G}_T(t)\ddot{G}_T(u)\}dudt. \qquad (C.29)$$

All of the terms involving a second derivative of the gating function cancel and leave

$$E[(M_2 - \hat{P}B_T^2)^2] = (2\pi)^{-4}T^{-2} \int_t \int_u G_T(t)G_T(u)$$

$$\cdot \ \{[\dot{R}_\xi(0)\dot{R}_\xi^*(0) + \dot{R}(t - u)\dot{R}_\xi^*(t - u)]$$

$$\cdot \ \dot{G}_T(t)\dot{G}_T(u)$$

$$+ \ [\ddot{R}_\xi(0)\dot{R}_\xi^*(0) + \dot{R}(t - u)\ddot{R}_\xi^*(t - u)]$$

$$\cdot \ G_T(t)\dot{G}_T(u)$$

$$+ \ [\dot{R}_\xi(0)\ddot{R}_\xi^*(0) + \ddot{R}_\xi(t - u)\dot{R}_\xi^*(t - u)]$$

$$\cdot \ \dot{G}_T(t)G_T(u)$$

$$+ \ [\ddot{R}_\xi(0)\ddot{R}_\xi^*(0) + \ddot{R}_\xi(t - u)\ddot{R}_\xi^*(t - u)]$$

$$\cdot \ G_T(t)G_T(u)\}dudt. \qquad (C.30)$$

The first half of each of the first three terms integrate to zero because of symmetry. The second half of each of those terms is zero when they are put in terms of $S_\xi(f)$ and when large T is assumed. Therefore, for large T

$$E[(M_2 - \hat{P}B_T^2)^2] = P^2[B^2 + B_\rho^2 + f_a^2] + T^{-1} \int_f f^4 S_\xi^2(f)df. \tag{C.31}$$