CHAPTER II

THE ESTIMATOR STRUCTURE

It is desirable to obtain the optimum estimators of the PSD shape parameters. If the mean square error is chosen as a measure of the quality of an estimator, then the estimators that are sought for f_a and B^2 must minimize this error.

Recall the parameter definitions:

$$f_a \equiv \frac{\int_f fS(f)df}{\int_f S(f)\, df} \qquad (2.1)$$

$$B^2 \equiv \frac{\int_f (f - f_a)^2 S(f)df}{\int_f S(f)\, df} \qquad (2.2)$$

Hence, the estimators $\hat{f}_a$ and $\hat{B}^2$ (of f_a and B^2, respectively) are optimum when the quantities

$$\epsilon(\hat{f}_a) = E[(f_a - \hat{f}_a)^2] \qquad (2.3)$$

and

$$\epsilon(\hat{B}^2) = E[(B^2 - \hat{B}^2)^2] \qquad (2.4)$$

are minimized (where the symbol E[ ] denotes the ensemble average of the quantity within the brackets).

Optimum Estimation

It is shown in this section that the method used to calculate the estimators can be restricted, without loss of generality, to a particular class. The class of parameter estimators used is that class which attempts to estimate S(f) first and then use this result and the definition of the parameter to calculate the parameter estimate.

In the case of a power spectrum whose shape is known, $\hat{f}_a$'s mean sqaure error has been minimized (8). In this investigation, the power spectrum is an unknown. Hence, minimization of the mean square error for each estimator requires the simultaneous estimation of all of the independent parameters of the PSD that can be determined after the process has passed through a window function. If T is the width of the finite time window employed in taking the measurement (i.e., the observation time), then the number of independent parameters, N, which specify the waveform is proportional to BT (Sampling Theorem).

Assume that there exists a particular N-set of independent PSD shape parameters which generates the optimum estimator for either f_a or B^2. Since a one to one mapping from the N-space of PSD shape parameters to any other N-space of PSD shape parameters may be defined, any N independent PSD shape parameters may be utilized in the estimation procedure. The set of N independent values of S(f) is an N-set of parameters which simplifies the following work. Therefore, a solution of the original problem is obtained by an estimation of S(f) that minimizes error in the computation of the PSD shape parameter estimates $\hat{f}_a$ and $\hat{B}^2$ from $\hat{S}(f)$.

Mean Square Error of $\hat{f}_a$. Recall that the quality of the spectral estimate is measured by the error incurred in the computation of $\hat{f}_a$ and $\hat{B}^2$ from $\hat{S}(f)$ (not the mean square error of $\hat{S}(f)$). The mean square error in $\hat{S}(f)$ as a function of frequency is

$$\sigma_s^2(f) \equiv E\{[S(f) - \hat{S}(f)]^2\}. \tag{2.5}$$

Hence, a relationship between the errors $\epsilon(\hat{f}_a)$ and $\sigma_s^2(f)$ must be found. The results of the preceding section and the definition of f_a yield

$$\hat{f}_a = \frac{\int_f f\hat{S}(f)df}{\int_f \hat{S}(f)\, df}. \tag{2.6}$$

Then the relationship is obtained by considering

$$\epsilon(f_a) = E\left\{\left[\frac{\int_f fS(f)df}{\int_f S(f)\, df} - \frac{\int_f f\hat{S}(f)df}{\int_f \hat{S}(f)\, df}\right]\right\}. \tag{2.7}$$

The quantity within the square brackets can be put over a common denominator and rewritten as:

$$\frac{\left[\int_f S(f)df\right]\left[\int_f f\hat{S}(f)df\right] - \left[\int_f \hat{S}(f)df\right]\left[\int_f fS(f)df\right]}{\left[\int_f \hat{S}(f)df\right]\left[\int_f S(f)df\right]} \tag{2.8}$$

Since a product of two integrals may be expressed as a double integral, this can be rewritten as

$$\frac{\int_f \int_g f\hat{S}(f)S(g)\ dgdf - \int_f \int_g g\hat{S}(f)S(f)\ dgdf}{\int_f \int_g \hat{S}(f)S(g)\ dgdf}$$

$$= \frac{\int_f \int_g (f - g)\hat{S}(f)S(g)\ dgdf}{\int_f \int_g \hat{S}(f)S(g)\ dgdf}. \qquad (2.9)$$

Letting

$$\hat{S}(f) = S(f) + \Delta s(f), \qquad (2.10)$$

(2.8) becomes

$$\frac{\left[\int_f \int_g (f - g)S(f)S(g)\ dgdf + \int_f \int_g (f - g)\Delta s(f)S(g)\ dgdf\right]}{\int_f \int_g S(f)S(g)\ dgdf + \int_f \int_g \Delta s(f)S(g)\ dgdf} \qquad (2.11)$$

Evaluating the first integral in the numerator:

$$\int_f \int_g (f - g)S(f)S(g)\ dgdf$$

$$= \int_f S(f)[f \int_g S(g)\ dg - \int_g gS(g)dg]df$$

$$= \int_f S(f)[fP - f_a P]df$$

$$= P[\int_f fS(f)\ df - f_a \int_f S(f)\ df]$$

$$= P[f_a P - f_a P]$$

$$= 0. \qquad (2.12)$$

Therefore,

$$\epsilon(\hat{f}_a) = E\left\{\left[\frac{\int_f \int_g (f - g)\Delta s(f)S(g)\ dgdf}{P^2 + P\int_f \Delta s(f)\ df}\right]^2\right\}. \qquad (2.13)$$

The denominator can be expressed as a power series.

$$\epsilon(\hat{f}_a) = E\left\{\left[\frac{\int_f \int_g (f - g)S(f)S(g)\ dgdf}{P^2}\right.\right.$$

$$\left.\left.\cdot \sum_{r=0}^{\infty}\left[\frac{\int_f \Delta s(f)\ df}{-P}\right]^r\right]^2\right\}. \qquad (2.14)$$

Since $\int_f \Delta s(f)df$, the error in the estimate of P, is much less than P for large T (see Appendix D), the sum may be approximated by its first term. Therefore,

$$\epsilon(\hat{f}_a) \approx P^{-4}E\left\{\left[\int_f \int_g (f - g)\Delta s(f)S(g)\ dgdf\right]^2\right\}. \quad (2.15)$$

The integral can be partially evaluated.

$$\epsilon(\hat{f}_a) = p^{-4}E\left\{\left[P \int_f (f - f_a)\Delta s(f)\ df\right]^2\right\}. \quad (2.16)$$

Or,

$$\epsilon(\hat{f}_a)$$

$$= (1/P^2)E\left[\int_f \int_g (f - f_a)(g - f_a)\Delta s(f)\Delta s(g)\right] dgdf. \quad (2.17)$$

Bringing the expected value into the integral results in a factor

$$R_s(f,g) \equiv E[\Delta s(f), \Delta s(g)] \quad (2.18)$$

which is the correlation of two adjacent values of the spectral estimator. At $f = g$ this quantity is $\sigma_s^2(f)$, the variance of the spectral estimator as a function of frequency, and when f is "close" to g the function is relatively constant. But as $f - g$ increases, $R_s(f - g)$ rapidly goes to zero. Let f_c, the correlation frequency, be that frequency differential at which the two samples are first said to be uncorrelated, then f_c is of order of magnitude T^{-1}. Therefore, for large T, it is reasonable to make the approximation

$$R_s(f,g) \approx \sigma_s^2(f)G_{f_c}(f - g) \quad (2.19)$$

Using this approximation, the double integral can be reduced to a single integral.

$$\epsilon(\hat{f}_a) \approx (1/P^2) \int_f (f - f_a)\sigma_s^2(f) \cdot \int_{f-f_c}^{f+f_c} (g - f_a)dgdf. \quad (2.20)$$

Integrating with respect to g,

$$\epsilon(\hat{f}_a) = (2f_c/P^2) \int_f (f - f_a)^2\sigma_s^2(f)\ df. \quad (2.21)$$

Therefore, since f_c is inversely proportional to T,

$$\epsilon(\hat{f}_a) = (k_1/P^2T) \int_f (f - f_a)^2\sigma_s^2(f)\ df \quad (2.22)$$

where k_1 is dependent somewhat on the shape of the window function (as a function of time) used in calculating $\hat{S}(f)$. To aid in the analysis of this result, let $u = (f - f_a)$ and then let $f = u$. This results in

$$\epsilon(\hat{f}_a) = (k_1/P^2T) \int_f f^2\ \sigma_s^2(f + f_a)\ df. \quad (2.23)$$

<u>Mean Square Error of $\hat{B}^2$</u>. The calculation of $\hat{B}^2$'s mean square error is analogous to the computation of $\hat{f}_a$'s mean square error. Recall that the spectrum can be estimated first without any loss of generality. Then, from the definition of $\hat{B}^2$,

$$\epsilon(\hat{B}^2) = E\left\{\left[\frac{\int_f f^2 S(f)df}{\int_f S(f)\ \ df} - \left[\frac{\int_f fS(f)df}{\int_f S(f)\ df}\right]\right.\right.$$

$$\left.\left. - \frac{\int_f f^2\hat{S}(f)df}{\int_f \hat{S}(f)\ \ df} + \left[\frac{\int_f f\hat{S}(f)df}{\int_f \hat{S}(f)\ df}\right]^2\right]^2\right\}. \tag{2.24}$$

After placing the fractions over a common denominator and writing the product of integrals as a multiple integral, this expression becomes

$$\epsilon(\hat{B}^2) = E\left\{\left[\frac{\left(\int_f\int_g\int_u\int_v S(f)S(g)\hat{S}(u)\hat{S}(v) \cdot [v^2 - uv - f^2 + fg]dvdudgdf\right)}{\int_f\int_g\int_u\int_v S(f)S(g)\hat{S}(u)\hat{S}(v)dvdudgdf}\right]^2\right\} \tag{2.25}$$

Again, let $\hat{S}(f) = S(f) + \Delta s(f)$. Then,

$$\epsilon(\hat{B}^2) = E\left\{\left[\frac{\left(\int_f\int_g\int_u\int_v [v^2 - uv - f^2 + fg]S(f)S(g) \cdot [S(u)S(v) + S(u)\Delta s(v) + \Delta s(u)S(v) + \Delta s(u)\Delta s(v)]dvdudgdf\right)}{\left(\int_f\int_g\int_u\int_v S(f)S(g) \cdot [S(u)S(v) + S(u)\Delta s(v) + \Delta s(u)S(v) + \Delta s(u)\Delta S(v)]dvdudgdf\right)}\right]^2\right\} \tag{2.26}$$

Since, on the average, $\Delta s(f) \ll S(f)$, the second order terms in both the numerator and denominator can be neglected in the case of long time averages. Also, the integral corresponding to the first term of the third factor in the numerator is zero for any $S(f)$. Therefore,

$$\epsilon(\hat{B}^2) = E\left\{\left[[1/(P^4 + 2P^3 \int_f S(f)df)] \int_f \int_g \int_u \int_v S(f)S(g) \cdot [v^2 - uv - f^2 + fg] \cdot [S(u)\Delta s(v) + \Delta s(u)S(v)]dvdudgdf\right]^2\right\}. \qquad (2.27)$$

Again, the denominator can be expressed as an alternating power series which occurs in the numerator. But, all of the terms, except for the first term, contribute second order effects or less. Therefore, the error may be further approximated as

$$\epsilon(\hat{B}^2) \approx P^{-8}E\left\{\left[\int_f \int_g \int_u \int_v [v^2 - uv - f^2 + fg]S(f)S(g) \cdot [S(u)\Delta s(v) + \Delta s(u)S(v)]dvdudgdf\right]^2\right\}. \qquad (2.28)$$

The integrations over f and g can be performed by using the following identities:

$$\int_f S(f)\, df = P,$$

$$\int_f fS(f)\, df = f_a P,$$

$$\int_f f^2 S(f)\ df = [B^2 + f_a^2]P. \qquad (2.29)$$

Therefore,

$$\epsilon(\hat{B}^2) = P^{-4}E\left\{\left[\int_u \int_v [v^2 - uv - B^2] \cdot [S(u)\Delta s(v) + \Delta s(u)S(v)]dvdu\right]^2\right\}. \qquad (2.30)$$

Writing the product of integrals as a multiple integral and bringing the expected value inside the integral yields

$$\epsilon(\hat{B}^2) = P^{-4}\int_f \int_g \int_u \int_v [g^2 - fg - B^2][v^2 - uv - B^2] \cdot \{S(f)S(u)E[\Delta s(g)\Delta s(v)] + S(f)S(v)E[\Delta s(g)\Delta s(u)] + S(g)S(u)E[\Delta s(f)\Delta s(v)] + S(g)S(v)E[\Delta s(f)\Delta s(u)]\}dvdudgdf. \qquad (2.31)$$

Two of the integrations can be performed for each of the terms in the third factor.

$$\epsilon(\hat{B}^2) = P^{-2}\left\{\int_g \int_v [g^2 - f_a g - B^2][v^2 - f_a v - B^2] \cdot E[\Delta s(g)\Delta s(v)]dvdg + \int_g \int_u [g^2 - f_a g - B^2][f_a^2 - uf_a]E[\Delta s(g)\Delta s(u)]duds + \int_f \int_v [f_a^2 - f_a f][v^2 - f_a v - B^2]E[\Delta s(f)\Delta s(v)]dvdf\right.$$

$$+ \int_f \int_u [f_a^2 - f_a f][f_a^2 - f_a u]E[\Delta s(f)\Delta s(u)]dudf\Big\}. \tag{2.32}$$

Changing the name of the dummy variables in three of the integrals to correspond with the first integral yields

$$\begin{aligned}\epsilon(\hat{B}^2) = P^{-2} \int_g \int_v E[\Delta s(g)\Delta s(v)]\Big\{[g^2 - f_a g - B^2] \\ \cdot [v^2 - 2f_a v + f_a^2 - B^2] \\ + [f_a^2 - f_a g][v^2 - 2f_a v + f_a^2 - B^2]\Big\}dvdg.\end{aligned} \tag{2.33}$$

That is,

$$\begin{aligned}\epsilon(\hat{B}^2) = P^{-2} \int_g \int_v E[\Delta s(g)\Delta s(v)][g^2 - 2f_a g + f_a^2 - B^2] \\ \cdot [v^2 - 2f_a v + f_a^2 - B^2]dvdg.\end{aligned} \tag{2.34}$$

As in the case of the derivation for $\epsilon(\hat{f}_a)$, let

$$E[\Delta s(g)\Delta s(v)] \approx \sigma_s^2(g)G_{f_c}(g - v) \tag{2.35}$$

for long time averages. Therefore,

$$\begin{aligned}\epsilon(\hat{B}^2) \approx P^{-2} \int_g \Big\{[g^2 - 2f_a g - f_a^2 - B^2]\sigma_s^2(g) \\ \cdot \int_{g-f_c}^{g+f_c} [v^2 - 2f_a v + f_a^2 - B^2]dv\Big\} dg\end{aligned} \tag{2.36}$$

and integrating with respect to v

$$\epsilon(\hat{B}^2) = (2f_c/P^2) \int_g [g^2 - 2f_a g - f_a^2 - B^2]\sigma_s^2(g) \cdot [g^2 + (1/3)f_c^2 - 2f_a g + f_a^2 - B^2]dg. \quad (2.37)$$

Or,

$$\epsilon(\hat{B}^2) = (2f_c/P^2) \int_g [(g - f_a)^2 - B^2]^2\sigma_s^2(g)\, dg + (f_c^2/(3P^2)) \int_g [(g - f_a)^2 - B^2]^2\sigma_s^2(g)dg. \quad (2.38)$$

But, f_c is inversely proportional to T, the observation time. Therefore, the second term can be ignored for large T and

$$\epsilon(\hat{B}^2) = k_2/(P^2T) \int_g [(g - f_a)^2 - B^2]^2\sigma_s^2(g)\, dg. \quad (2.39)$$

By the change of variables

$$f = g - f_a,$$

the error becomes

$$\epsilon(\hat{B}^2) = k_2/(P^2T) \int_f (f^2 - B^2)^2\sigma_s^2(f + f_a)df. \quad (2.40)$$

The result for f_a is repeated here for convenience:

$$\epsilon(\hat{f}_a) = k_1/(P^2T) \int_f f^2\sigma_s^2(f + f_a)\, df.$$

These formulas give a criterion by which to judge a proposed estimator for S(f). If an estimator can be found which minimizes $\sigma_s^2(f)$ at each value of f, then it is clearly also optimum for the estimation of f_a and B^2. But, if there is an interaction between PSD estimates at different frequencies, then those estimates at frequencies far from f_a should be given a priority that is higher than that of those estimates at frequencies close to f_a. This is due to the fact that the two weighting functions f^2 and $(f^2 - B^2)^2$ are small only in a range around zero.

The Periodogram

The narrow band characteristics of the reflected signal in radar meteorology makes the periodogram, a classically used spectral estimator, a good choice. That is, the periodogram's mean square error at each frequency is proportional to the square of the true value of the spectrum. Hence, both $\epsilon(\hat{f}_a)$ and $\epsilon(\hat{B}^2)$ are small when the periodogram is used for $\hat{S}(f)$. This confirms the validity of the choice in this case.

The periodogram, as used here, is defined to be

$$\hat{S}(f) \equiv \frac{1}{T}|Z_T(2\pi f)|^2 \qquad (2.41)$$

where

$$Z_T(2\pi f) = \int_f z(t)p(t)^{-j2\pi ft}\, dt \qquad (2.42)$$

and p(t) is a real, time limited, window function for the measurement. Again, z(t) is a particular sample function of the process's complex envelope referenced to the transmitted frequency.

Miller and Rochwarger (7) show that if f'_a and $B^{2\prime}$ are the power mean frequency and mean square bandwidth, respectively, of the process after passing through the time window, then

$$f_a = f'_a - f_p \tag{2.43}$$

and

$$B^2 = B^{2\prime} - B_p^2 \tag{2.44}$$

where f_p is the mean frequency of the energy spectrum, $|P(2\pi f)|^2$, for the window function and B_p^2 is the mean square bandwidth of $|P(2\pi f)|^2$. Since p(t) is real, f_p is zero and the effect of the window function on $\hat{f}_a$ is to change the error in the estimate. But, B_p^2 is not zero and introduces a known bias in $\hat{B}^2$. This bias can be eliminated. A problem, however, occurs because of the time limited constraint on p(t). This makes B_p^2 infinite and care must be taken in the elimination of this bias.

To facilitate the following work, the window function, p(t), is normalized so that if T is the finite width of p(t), then

$$\int_t p^2(t)dt = T \tag{2.45}$$

The window function is also separated into two factors, $\rho(t)$ and $G_T(t)$. The first factor, $\rho(t)$, is a smooth absolutely integrable function on $(-\infty,\infty)$ and has a finite mean square spectral bandwidth, B_ρ^2. $G_T(t)$ takes into account the time limited characteristic. Therefore, if

$$\xi(t) = Z(t)\rho(t), \tag{2.46}$$

then

$$Z_T(2\pi f) = \int_t \xi(t)G_T(t)e^{-j2\pi ft}dt. \tag{2.47}$$

Thus, the effects of the window function can be included in all of the following work.

The Estimator Structure

Formulas for the parameter estimates can be derived directly by beginning with the definition for each parameter and substituting the periodogram for the power spectrum. Again, care must be taken in the case of $\hat{B}^2$ to remove the bias introduced by the window function.

The Mean Frequency Estimator. Recall the definition for f_a. Then, after substitution of the periodogram,

$$\hat{f}_a = \frac{T^{-1}\int_f f|Z_T(2\pi f)|^2\, df}{T^{-1}\int_f |Z_T(2\pi f)|^2\, df}. \tag{2.48}$$

The denominator can be recognized as $\hat{P}$, an estimator for the total power in the process; the numerator is $\widehat{f_aP}$, an estimator for the product of the total power with the power mean frequency. Hence, time domain expressions for $\hat{P}$ and $\widehat{f_aP}$ are necessary.

To obtain a time domain expression for $\hat{P}$, simply substitute the Fourier transform of $\xi(t)G_T(t)$ for $Z_T(2\pi f)$. Therefore,

$$\hat{P} = T^{-1} \int_f \left[\int_t \xi(t)G_T(t)e^{-j2\pi ft}\, dt\right] \cdot \left[\int_\tau \xi^*(\tau)G_T(\tau)e^{j2\pi f\tau}\, d\tau\right]df. \qquad (2.49)$$

Changing the order of integration yields

$$\hat{P} = T^{-1} \int_t \int_\tau \xi(t)\xi^*(\tau)G_T(t)G_T(\tau) \int_f e^{j2\pi(\tau-t)f} df d\tau dt. \qquad (2.50)$$

Note that the integral over f is $\delta(\tau - t)$, since it is the inverse Fourier transform of a constant. Integration with respect to τ and use of the sifting property of impluse functions yields

$$\hat{P} = T^{-1} \int_t |\xi(t)|^2 G_T(t)\, dt. \qquad (2.51)$$

Or,

$$\hat{P} = T^{-1} \int_{-T/2}^{T/2} |\xi(t)|^2\, dt. \qquad (2.52)$$

The quadrature form of this result can now be stated. Let α be the real part of $\xi(t)$ and let β be the imaginary part of $\xi(t)$ where the independent variable, t, of α and β are assumed understood for convenience. Then

$$\hat{P} = T^{-1} \int_{-T/2}^{T/2} [\alpha^2 + \beta^2]dt. \qquad (2.53)$$

The same result appears in polar form as

$$\hat{P} = T^{-1} \int_{-T/2}^{T/2} A^2 \, dt \qquad (2.54)$$

where A is the instantaneous envelope of the process after the process has passed through the window.

A similar derivation can be done for the numerator. Substitution of the conjugate of the Fourier transform of Z(t)p(t) for $Z_T^*(2\pi f)$ yields

$$\widehat{Pf_a} = T^{-1} \int_f f[\int_t \xi^*(t)G_T(t)e^{j2\pi ft}dt]Z_T(2\pi f)df. \qquad (2.55)$$

Interchanging the order of integration yields

$$\widehat{Pf_a} = -j(2\pi T)^{-1} \int_t \xi^*(t)G_T(t) \int_f j2\pi fZ_T(2\pi f)e^{j2\pi ft}dfdt. \qquad (2.56)$$

The inner integral can be recognized as the derivative of the product $\xi(t)G_T(t)$. Therefore,

$$\widehat{Pf_a} = -j(2\pi T)^{-1} \int_t \xi^*(t)G_T(t)[\dot{\xi}(t)G_T(t) + \xi(t)\dot{G}_T(t)]dt. \qquad (2.57)$$

The quadrature form can be obtained by replacing $\xi(t)$ by its real and imaginary parts, α and β. Note that

$$\dot{G}_T(t) = \delta(t + T/2) - \delta(t - T/2) \qquad (2.58)$$

where $\delta(t)$ is the Dirac delta function. Hence, the numerator is

$$\widehat{Pf_a} = -j/(2\pi T) \int_t \{[\alpha\dot{\alpha} + \beta\dot{\beta}] + [\alpha^2 + \beta^2]$$

$$\cdot [\delta(t + T/2) - \delta(t - T/2)]$$

$$+ j[\alpha\dot{\beta} - \beta\dot{\alpha}] + j[\alpha\beta - \beta\alpha]$$

$$\cdot [\delta(t + T/2) - \delta(t - T/2)]\}G_T(t)dt. \quad (2.59)$$

The first and second terms cancel when integrated since

$$\int_t [\alpha\dot{\alpha} + \beta\dot{\beta}]G_T(t)dt = [\alpha^2/2 + \beta^2/2]\Big|_{t = -T/2}^{t = T/2} \quad (2.60)$$

and

$$\int_t [\alpha^2 + \beta^2][\delta(t + T/2) - \delta(t - T/2)]G_T(t)\ dt$$

$$= 1/2[\alpha^2 + \beta^2]\Big|_{t = T/2.}^{t = -T/2} \quad (2.61)$$

The fourth term is identically zero. This leaves

$$\widehat{Pf_a} = (2\pi T)^{-1} \int_{-T/2}^{T/2} [\alpha\dot{\beta} - \beta\dot{\alpha}]dt. \quad (2.62)$$

Therefore,

$$\hat{f}_a = (2\pi)^{-1} \frac{\int_{-T/2}^{T/2} [\alpha\dot{\beta} - \beta\dot{\alpha}]dt}{\int_{-T/2}^{T/2} [\alpha^2 + \beta^2]dt}. \quad (2.63)$$

In the case of $\rho(t) = 1$, this result is identical to the power mean frequency estimator suggested by Bello (3). A block diagram for this measurement system is shown in Figure 2.

Some insight into this result is gained when the kernel of the numerator integral is rewritten as

$$[\alpha^2 + \beta^2] \; \frac{1}{1 + (\beta/\alpha)^2} \cdot \frac{\alpha\dot{\beta} - \beta\dot{\alpha}}{\alpha^2} \; . \tag{2.64}$$

The first factor is A^2, the instantaneous power; the second factor is ω_i, the instantaneous frequency, where

$$\begin{aligned} \omega_i &= \frac{d}{dt} [\theta(t)] \\ &= \frac{d}{dt} [\tan^{-1}(\beta/\alpha)]. \end{aligned} \tag{2.65}$$

The estimator then has the intuitively consistent polar form

$$\widehat{Pf_a} = (2\pi T)^{-1} \int_{-T/2}^{T/2} A^2 \omega_i \, dt. \tag{2.66}$$

That is,

$$\widehat{Pf_a} = T^{-1} \int_{-T/2}^{T/2} A^2 f_i \, dt. \tag{2.67}$$

Note that A^2 and f_i are slowly varying functions of time. Hence, weighting the instantaneous frequency by the power at each instant of time and then averaging over time yields the intuitively expected result of total power times f_a. A block diagram implementation of this envelope-frequency (or, polar form) is shown in Figure 3.

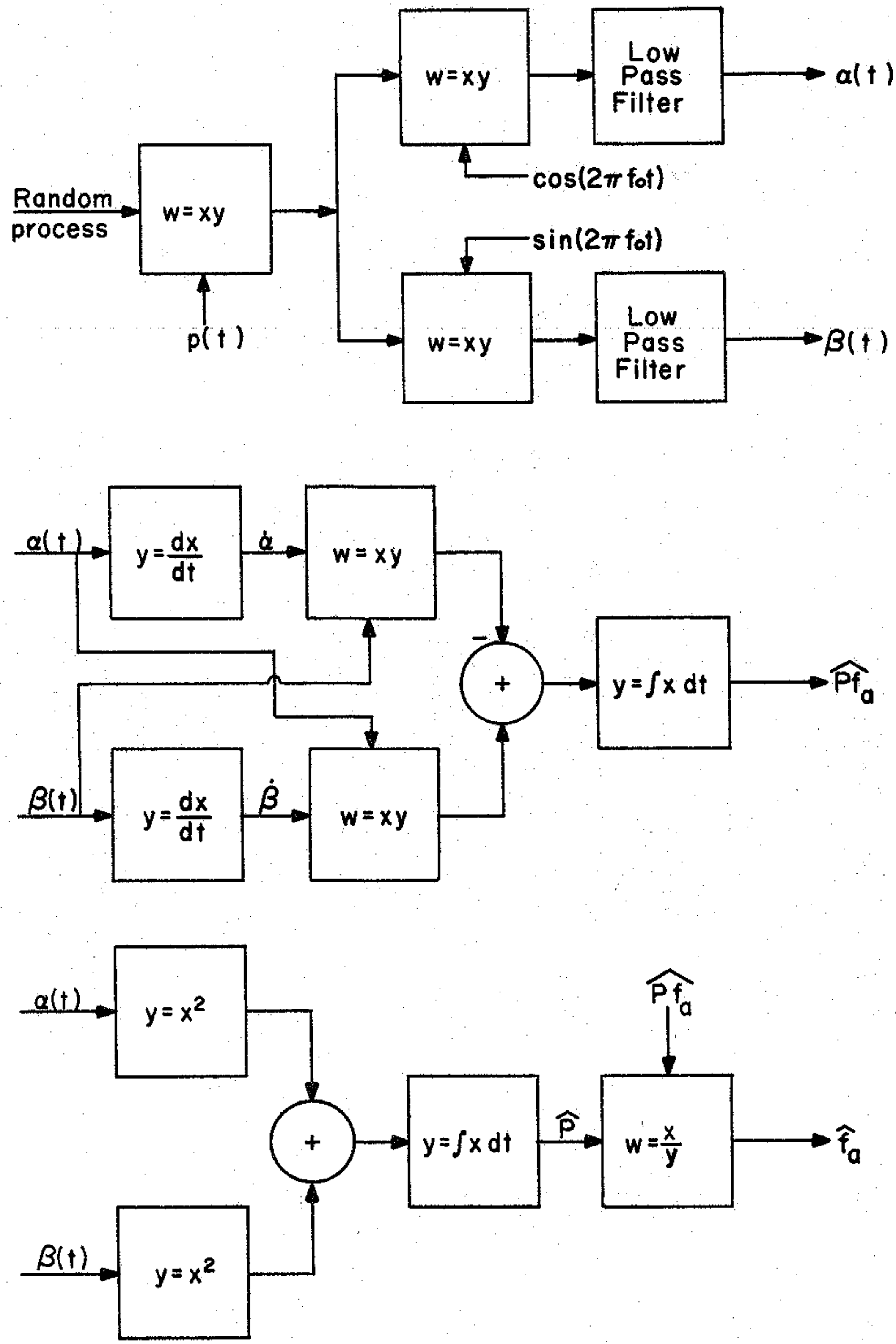

FIG. 2 The Power Mean Frequency Estimator: Quadrature Form

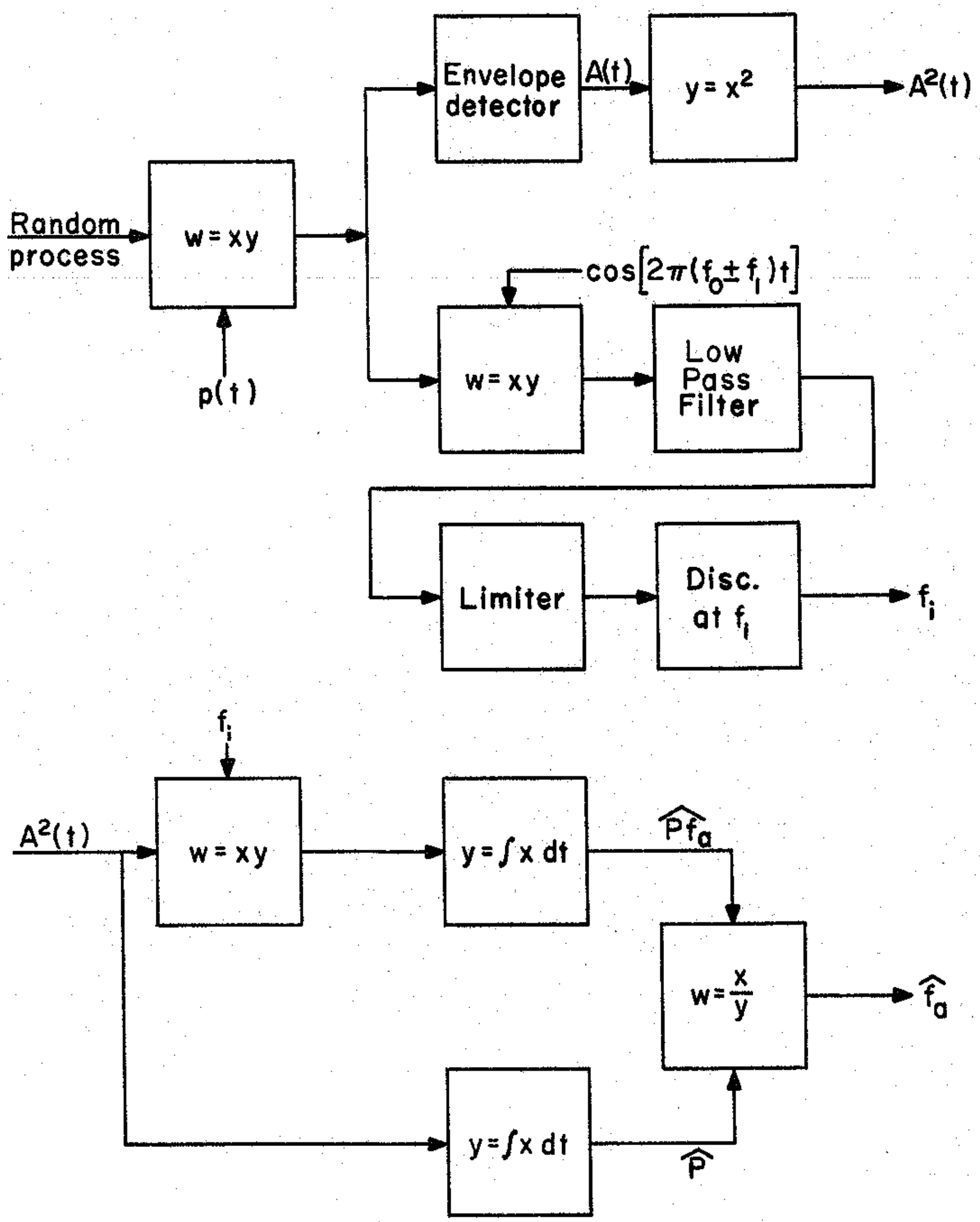

FIG. 3 The Power Mean Frequency Estimator: Polar Form

The Mean Square Bandwidth Estimator. The derivation of $\hat{B}^2$ is analogous to that for $\hat{f}_a$. In this case, however, an additional term must be added to the definition to remove the bias introduced by the window function. The window function consists of two factors

$$p(t) = \rho(t) G_T(t).$$

Since $\rho(t)$ is a smooth, absolutely integrable function, the spreading of the spectrum, B_ρ^2, induced by $\rho(t)$ is finite. Therefore, B_ρ^2 can be subtracted after the estimator is found. But, B_T^2, which is introduced by the discontinuities, is not finite and must be subtracted early. Therefore, before defining $\hat{B}^2$, the spreading of the spectrum due to $G_T(t)$ must be determined.

The average value of B_T^2 can be found from its definition:

$$\overline{B_T^2} = T^{-1} \int_f f^2 \left| \int_t G_T(t) e^{-j2\pi ft} \, dt \right|^2 df. \qquad (2.68)$$

Introducing a new variable τ and changing the order of integration yields

$$\overline{B_T^2} = -(4\pi^2 T)^{-1} \int_t \int_\tau G_T(t) G_T(\tau)$$

$$\cdot \int_f (j2\pi f)^2 e^{j2\pi(\tau - t)f} df d\tau dt. \qquad (2.69)$$

The integral over f is $\dot{\delta}(\tau - t)$. Therefore, integrating over τ,

$$\overline{B_T^2} = -(4\pi^2 T)^{-1} \int_t G_T(t)\ddot{G}_T(t)\ dt. \qquad (2.70)$$

Note that the average bandwidth spreading is due to the discontinuities and is, therefore, proportional, on a particular measurement, to the percentage of power in the process at the instant of each discontinuity. This suggests that the amount of bias to be subtracted is

$$B_T^2 = -(4\pi^2 T)^{-1} \int_t G_T(t)A^2\hat{P}^{-1}\ddot{G}_T(t)\ dt. \qquad (2.71)$$

Hence, $\hat{B}^2$ can be defined as

$$\hat{B}^2 = (T\hat{P})^{-1} \int_f f^2 |Z_T(2\pi f)|^2\ df - [\hat{f}_a]^2 - B_T^2 - B_\rho^2. \qquad (2.72)$$

The quantities $\hat{P}$ and $\hat{f}_a$ have already been discussed. B_ρ^2 is, by definition,

$$B_\rho^2 = \frac{\int_f f^2 \left| \int_t \rho(t) e^{-j2\pi ft}\ dt \right|^2 df}{\int_t \rho^2(t)\ dt} \qquad (2.73)$$

and, for any given window function, can be computed and used in the estimator.

Let

$$N = T^{-1} \int_f f^2 Z_T(2\pi f) Z_T^*(2\pi f)\ df - \hat{P}B_T^2. \qquad (2.74)$$

Consideration must now be given to finding a time domain expression for N. This can be done by substituting the conjugate of the Fourier transform of $\xi(t)G_T(t)$ for $Z_T(2\pi f)$. Then, changing the order of integration yields

$$N = -(4\pi^2 T)^{-1} \int_t \xi^*(t)G_T(t) \int_f (j2\pi f)^2 Z_T(2\pi f)e^{j2\pi ft}\,dfdt$$

$$+ (4\pi^2 T)^{-1} \int_t G_T(t)A^2\ddot{G}_T(t)\,dt. \qquad (2.75)$$

Or, integrating over f in the first integral and combining terms,

$$N = -(4\pi^2 T)^{-1} \int_t G_T(t)\{\xi^*(t)\ddot{\xi}(t) + 2\xi^*(t)\dot{\xi}(t)\dot{G}_T(t)$$

$$+ \xi^*(t)\xi(t)\ddot{G}_T(t) - A^2\ddot{G}_T(t)\}dt. \qquad (2.76)$$

Changing to the quadrature form:

$$N = -(4\pi^2 T)^{-1} \int_t G_T(t)\{[\alpha\ddot{\alpha} + \beta\ddot{\beta}] + 2[\alpha\dot{\alpha} + \beta\dot{\beta}]G_T(t)$$

$$+ j[\alpha\ddot{\beta} - \beta\ddot{\alpha}] + 2j[\alpha\dot{\beta} - \beta\dot{\alpha}]\dot{G}_T(t)\}dt. \qquad (2.77)$$

N can be simplified further through integration by parts. In the imaginary part, this results in

$$\mathscr{I}[N] = -(4\pi^2 T)^{-1}\{[\alpha\dot{\beta} - \beta\dot{\alpha}]\Big|_{t=-T/2}^{t=T/2} - \int_{-T/2}^{T/2} [\dot{\alpha}\dot{\beta} - \dot{\beta}\dot{\alpha}]dt$$

$$+ 2\int_t G_T(t)[\alpha\dot{\beta} - \beta\dot{\alpha}][\delta(t + T/2) - \delta(t - T/2)]dt\}. \qquad (2.78)$$

Or, noting a factor of one half in the sifting property of Dirac delta functions at the limit of an integral,

$$\mathscr{I}[N] = 0.$$

The same method used for the real part yields

$$\mathscr{R}[N] = -(4\pi^2 T)^{-1}\{[\alpha\dot{\alpha} + \beta\dot{\beta}]\Big|_{t\,=\,/T/2}^{t\,=\,T/2}$$

$$- \int_{-T/2}^{T/2} [\dot{\alpha}^2 + \dot{\beta}^2]\, dt \qquad (2.79)$$

$$+ 2 \int_t [\alpha\alpha + \beta\beta][\delta(t + T/2) - \delta(t - T/2)]\, dt\}.$$

The first and third terms cancel to leave

$$\mathscr{R}[N] = (4\pi^2 T)^{-1} \int_{-T/2}^{T/2} [\dot{\alpha}^2 + \dot{\beta}^2]\, dt. \qquad (2.80)$$

Hence,

$$\hat{B}^2 = (4\pi^2)^{-1} \frac{\int_{-T/2}^{T/2} [\dot{\alpha}^2 + \dot{\beta}^2]\, dt}{\int_{-T/2}^{T/2} [\alpha^2 + \beta^2]\, dt} - B_\rho^2 - (\hat{f}_a)^2. \qquad (2.81)$$

This is proportional to Bello's estimator $\hat{D}_1^2$ for the mean square bandwidth when $\rho(t) = 1$. This implementation is shown in Figure 4.

The polar form for $\hat{B}^2$ is also of interest and can be derived from the quadrature form by substitution of

$$\alpha = A \cos[\theta(t)] \qquad (2.82)$$

and

$$\beta = A \sin[\theta(t)]. \qquad (2.83)$$

Recalling that

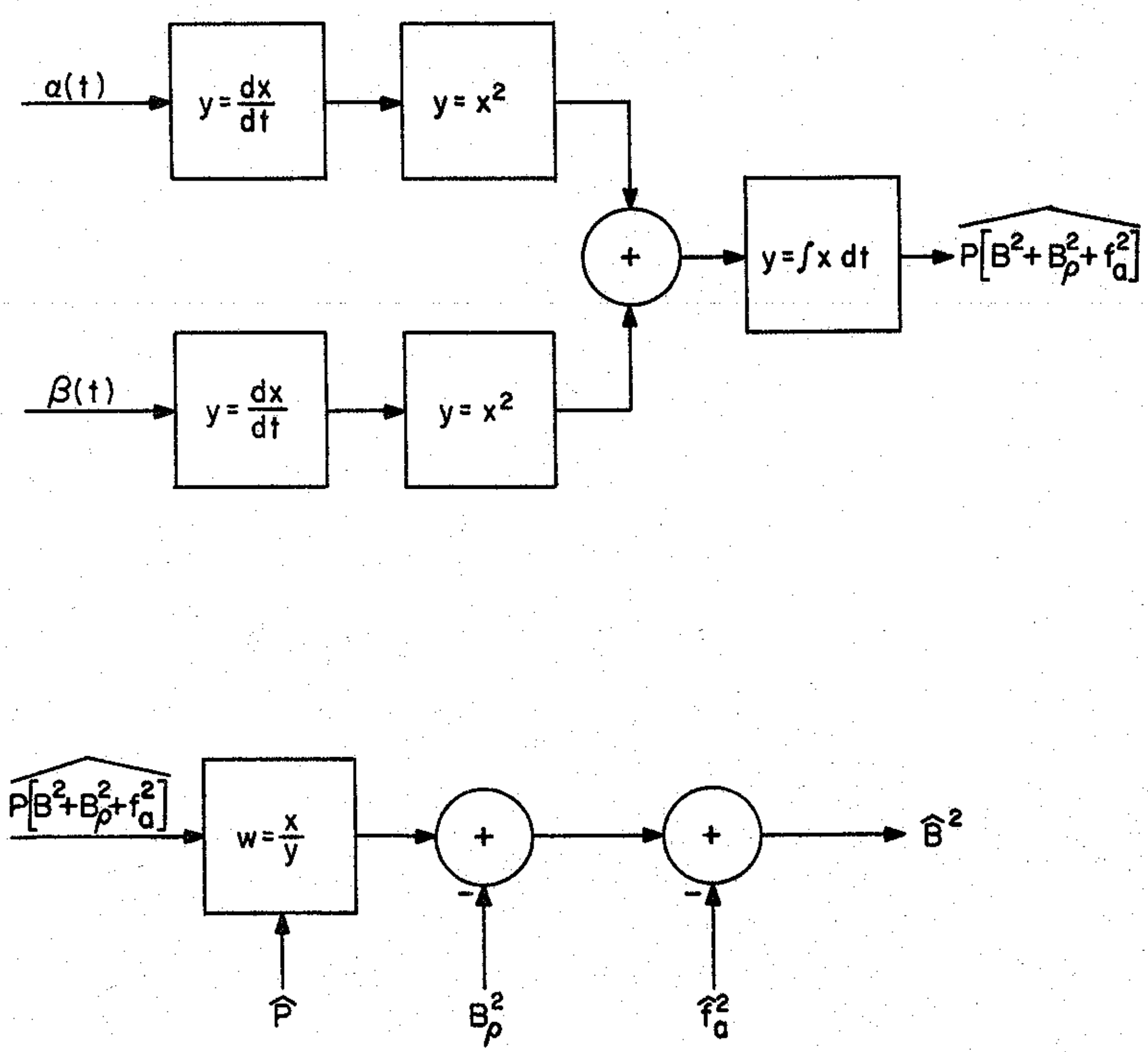

FIG. 4 The Mean Square Bandwidth Estimator: Quadrature Form

$$\dot{\theta}(t) = \omega_i, \tag{2.84}$$

the result is

$$\hat{B}^2 = (4\pi^2)^{-1} \frac{\int_{-T/2}^{T/2} [A^2\omega_i^2 + \dot{A}^2]\, dt}{\int_{-T/2}^{T/2} A^2\, dt} - B_\rho^2 - (f_a)^2. \tag{2.85}$$

This form can also be implemented (see Figure 5) and will give the same result as the quadrature form of the estimator.

Some intuitive analysis can be done on the polar form of $\hat{B}^2$. The first term in the numerator can be attributed to the quasi-stationary, random frequency modulation of the process. The second term is due to the random amplitude modulation and is proportional to Bello's estimator $\hat{D}_2^2$ in the special case of $\rho(t) = 1$ and a gaussian process.

These results indicate that $\hat{f}_a$ and $\hat{B}^2$ (or, Bello's estimators $\hat{f}_a$ and $\hat{D}_1$ for a rectangular window function) are consistent with the classical estimators when the periodogram is used for $\hat{S}(f)$. Therefore, a careful error analysis of the classical estimators is also applicable to these direct methods. Since $\hat{f}_a$ and $\hat{B}^2$ are obtained directly from the input data, they are easier to implement than the classical approaches. Hence, these direct estimators should be used whenever an explicit estimate of the power spectrum is unnecessary.

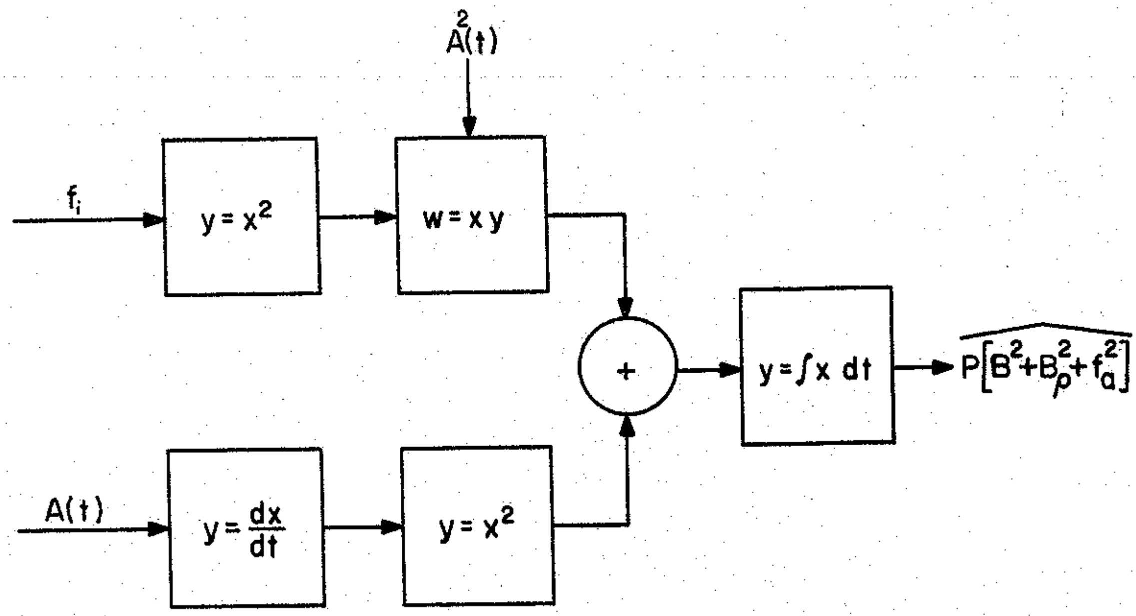

FIG. 5 The Mean Square Bandwidth Estimator: Polar Form