CHAPTER III

PROBABILISTIC ANALYSIS

In the last chapter, the estimators $\hat{f}_a$ and $\hat{B}^2$ are developed. These results are of value only if their statistical nature is known. Since these estimators are equivalent to those used classically, their statistics can be derived by investigating the behavior of the classical formulations of these estimators.

The Power Mean Frequency Estimator

To simplify the algebra involved in the following work, let

$$M_k = \int_f f^k \hat{S}(f)\, df \qquad (3.1)$$

and let

$$m_k = M_k - \overline{M}_k, \qquad (3.2)$$

where $\overline{M}_k$ is the average value of M_k. The estimator for f_a is then

$$\hat{f}_a = \frac{M_1}{M_0}. \qquad (3.3)$$

Or, in terms of the differential quantities m_0 and m_1,

$$\hat{f}_a = \frac{\overline{M}_1 + m_1}{\overline{M}_0 + m_0} \qquad (3.4)$$

Following the work of Miller and Rochwarger (7), the denominator can be replaced by its power series. Thus,

$$\hat{f}_a = (\overline{M}_1 + m_1)\frac{1}{\overline{M}_0}\sum_{k=0}^{\infty}\left(-\frac{m_0}{\overline{M}_0}\right)^k \qquad (3.5)$$

This series can be multiplied out and regrouped as

$$\hat{f}_a = \frac{\overline{M}_1}{\overline{M}_0} + \sum_{k=1}^{\infty}\left(-\frac{1}{\overline{M}_0}\right)^k\left[\frac{\overline{M}_1}{\overline{M}_0}(m_0)^k - m_1(m_0)^{k-1}\right] \qquad (3.6)$$

Since the error in M_0 and M_1 approaches zero asymptotically as T goes to infinity (see Appendix B), the quantity $\frac{m_0}{\overline{M}_0}$ is small for large T. Hence, the assumption of large T allows the approximation of $\hat{f}_a$ by the first few terms of the series above.

The number of terms used depends upon the degree of approximation desired. From Appendix D,

$$\overline{M}_0 = P$$

and

$$\overline{M}_1 = f_a P.$$

Thus, $\hat{f}_a$ can be written to a second order approximation as:

$$\hat{f}_a = f_a - P^{-1}[f_a m_0 - m_1] + P^{-2}[f_a m_0^2 - m_1 m_0] + O\left[\left(\frac{m_0}{P}\right)^3\right] \qquad (3.7)$$

The statistics of $\hat{f}_a$ can easily be determined from this result.

<u>Estimator Bias</u>. To determine the consistency of $\hat{f}_a$

it is necessary to find its average value. From equation (3.7)

$$E[\hat{f}_a] \approx f_a - P^{-1}[f_a E(m_0) - E(m_1)] + P^{-2}[f_a E(m_0^2) - E(m_0 m_1)]. \qquad (3.8)$$

Again, use of the results of Appendix D yields

$$E(m_k) = 0,$$

$$E(m_0^2) = T^{-1} \int_f S_\xi^2(f)\, df,$$

and

$$E(m_0 m_1) = T^{-1} \int_f f\, S_\xi^2(f)\, df,$$

where $S_\xi(f)$ is the power spectral density of the random process $\xi(t)$.

The mean value of $\hat{f}_a$ becomes

$$E(\hat{f}_a) \approx f_a + \frac{1}{P^2 T}\left[f_a \int_f S_\xi^2(f) df - \int_f f\, S_\xi^2(f) df\right]. \qquad (3.9)$$

This result can be simplified by making the change of variables $f + f_a \rightarrow f$ in both integrals. The second integral becomes

$$\int_f f\, S_\xi^2(f + f_a) df + f_a \int_f S_\xi^2(f + f_a) df.$$

Cancellation of terms yields the following as the final form:

$$E(\hat{f}_a) \approx f_a - \frac{1}{P^2 T} \int_f f\, S_\xi^2(f + f_a)df. \qquad (3.10)$$

The second term gives the bias of the estimator. It is independent of the value of f_a and is a function of only the power spectrum's shape. In the case of a symmetrical PSD, the bias is zero.

Hence, $\hat{f}_a$ is a biased estimator. The bias approaches zero asymptotically for large T; it is zero (at least to a second order approximation) for all T when the true spectrum is symmetrical.

Estimator Error. The measure of the "goodness" of the estimator is again

$$\epsilon[\hat{f}_a] = E[(f_a - \hat{f}_a)^2].$$

This expected error can be calculated for the first order approximation to $\hat{f}_a$ stated earlier:

$$\hat{f}_a \approx f_a - \frac{1}{P}[f_a m_0 - m_1].$$

Therefore,

$$(f_a - \hat{f}_a)^2 = \frac{1}{P^2}[f_a^2 m_0^2 - 2f_a m_0 m_1 + m_1^2] \qquad (3.11)$$

and

$$\epsilon(\hat{f}_a) = \frac{1}{P^2}[f_a^2 E(m_0^2) - 2f_a E(m_0 m_1) + E(m_1^2)]. \qquad (3.12)$$

Using the results of Appendix D,

$$\epsilon(\hat{f}_a) = \frac{1}{P^2T}\left[f_a^2 \int S_\xi^2(f)df - 2f_a \int f\ S_\xi^2(f) + \int f^2S_\xi^2(f)df\right]. \tag{3.13}$$

Or,

$$\epsilon(\hat{f}_a) = \frac{1}{P^2T}\int_f (f - f_a)^2S_\xi^2(f)df. \tag{3.14}$$

Employing the same change of variables as in the previous section,

$$\epsilon(\hat{f}_a) = \frac{1}{P^2T}\int_f f^2S_\xi^2(f + f_a)df. \tag{3.15}$$

Therefore, the mean square error of $\hat{f}_a$ is independent of the true value of f_a. It is a function of T, the width of the finite time window, and the width of the spectrum. This result agrees with that of Miller and Rochwarger (7) in the special case of a large signal to noise ratio.

The RMS Bandwidth Estimator

Using the notation of the previous section, the estimator for the square of the RMS bandwidth can be expressed as

$$\hat{B}^2 = \frac{M_2}{M_0} - (\hat{f}_a)^2 - B_\rho^2 - B_T^2. \tag{3.16}$$

Replacing $\hat{f}_a$ by its power series expansion and utilizing a similar power series for the quantity

$$\frac{M_2 - M_0B_T^2}{M_0} = \frac{\overline{M}_2 + m_2 - M_0B_T^2}{\overline{M}_0 + m_0} \tag{3.17}$$

the estimator becomes

$$\hat{B}^2 = \frac{\overline{(M_2 - M_0 B_T^2 + m_2)}}{\overline{M}_0} \sum_{k=0}^{\infty} \left(- \frac{m_0}{\overline{M}_0}\right)^k$$

$$- \left\{ \frac{\overline{M}_1}{\overline{M}_0} + \sum_{k=1}^{\infty} \left(- \frac{1}{\overline{M}_0}\right)^k \left[\frac{\overline{M}_1}{\overline{M}_0} (m_0)^k - m_1 (m_0)^{k-1} \right] \right\}^2 - B_\rho^2. \quad (3.18)$$

Or, after expanding both terms,

$$\hat{B}^2 = \frac{\overline{(M_2 - M_0 B_T^2)}}{\overline{M}_0} - \left(\frac{\overline{M}_1}{\overline{M}_0}\right)^2 - B_\rho^2$$

$$+ \sum_{k=1}^{\infty} \left(- \frac{1}{\overline{M}_0}\right)^k \left[\frac{\overline{(M_2 - M_0 B_T^2)}}{\overline{M}_0} (m_0)^k - m_2 (m_0)^{k-1} \right]$$

$$- 2\left(\frac{\overline{M}_1}{\overline{M}_0}\right) \sum_{k=1}^{\infty} \left(- \frac{1}{\overline{M}_0}\right)^k \left[\frac{\overline{M}_1}{\overline{M}_0} (m_0)^k - m_1 (m_0)^{k-1} \right] \quad (3.19)$$

$$- \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \left(- \frac{1}{\overline{M}_0}\right)^{\ell+k} \left[\left(\frac{\overline{M}_1}{\overline{M}_0}\right)^2 (m_0)^{\ell+k} - 2\left(\frac{\overline{M}_1}{\overline{M}_0}\right) m_1 (m_0)^{\ell+k-1} + (m_1)^2 (m_0)^{\ell+k-2} \right].$$

The higher order terms may again be neglected for large T. Thus, through terms of second order, the expression becomes:

$$\hat{B}^2 \approx \left[\frac{\overline{(M_2 - M_0 B_T^2)}}{\overline{M}_0} - \left(\frac{\overline{M}_1}{\overline{M}_0}\right)^2 - B_\rho^2 \right]$$

$$- \frac{1}{\overline{M}_0}\left[\frac{\overline{(M_2 - M_0 B_T^2)}}{\overline{M}_0} - 2\frac{\overline{M}_1^2}{\overline{M}_0^2} m_0 + 2\left(\frac{\overline{M}_1}{\overline{M}_0}\right) m_1 - m_2\right]$$

$$+ \frac{1}{\overline{M}_0^2}\left[\left(\frac{\overline{(M_2 - M_0 B_T^2)}}{\overline{M}_0} - 3\frac{\overline{M}_1^2}{\overline{M}_0^2}\right)(m_0)^2 + \left(4\frac{\overline{M}_1}{\overline{M}_0} m_1 - m_2\right)m_0 - (m_1)^2\right] \qquad (3.20)$$

Recalling that

$$E(M_1) = f_a P$$

and noting that

$$E[M_2 - M_0 B_T^2] = P[B^2 + B_\rho^2 + f_a^2], \qquad (3.21)$$

the bandwidth estimator can be rewritten as

$$\hat{B}^2 \approx B^2 - \frac{1}{P}\,[(B^2 - f_a^2)m_0 + 2f_a m_1 - m_2]$$
$$+ \frac{1}{P^2}\,[(B^2 - 2f_a^2)(m_0)^2 + 4f_a(m_0 m_1) - m_0 m_2 - (m_1)^2]. \qquad (3.22)$$

The statistics of $\hat{B}^2$ may be evaluated from this result.

<u>Estimator Bias</u>. Determination of the average value of the estimator shows its consistency or inconsistency. From (3.22),

$$E[\hat{B}^2] = B^2 - \frac{1}{P}\,[(B^2 - f_a^2)E(m_0) + 2f_a E(m_1) - E(m_2)]$$
$$+ \frac{1}{P^2}\,[(B^2 - 2f_a^2)E(m_0^2) + 4f_a E(m_0 m_1) - E(m_0 m_2)$$
$$- E(m_1^2)]. \quad (3.23)$$

Recalling the results of Appendix D,

$$E(\hat{B}^2) = B^2 + \frac{1}{P^2 T}[(B^2 - 2f_a^2)\int_f S_\xi^2(f)df + 4f_a \int_f fS_\xi^2(f)df$$
$$- 2\int_f f^2 S_\xi^2(f)df]. \quad (3.24)$$

Collecting terms,

$$E(\hat{B}^2) = B^2 + \frac{1}{P^2 T}\int_f [B^2 - 2(f - f_a)^2]S_\xi^2(f)df. \quad (3.25)$$

To explicitly express the variation of this form with f_a, let f be replaced by $f + f_a$ in each integral. Then,

$$E(\hat{B}^2) = B^2 + \frac{1}{P^2 T}\int_f (B^2 - 2f^2)S_\xi^2(f + f_a)df. \quad (3.26)$$

Therefore, $\hat{B}^2$ is also a biased estimator in the general case. As was the situation with $\hat{f}_a$, this estimator is asymptotically unbiased. The bias for $\hat{B}^2$ is also independent of the true value of f_a and only is a function of the spectrum's shape. The bias is zero for all T in the special case of a gaussian shaped spectrum.

<u>Estimator Error</u>. To utilize the criterion for the

"goodness" of $\hat{B}^2$ that was used for the case of $\hat{f}_a$ requires the use of the first order approximation for $\hat{B}^2$:

$$\hat{B}^2 \approx B^2 - \frac{1}{P} [(B^2 - f_a^2)m_0 + 2f_a m_1 - m_2]. \quad (3.27)$$

Therefore,

$$[B^2 - \hat{B}^2]^2 \approx \frac{1}{P^2} [(B^4 - 2f_a^2 B^2 + f_a^4)m_0 m_0 + 4f_a^2 m_1 m_1 + m_2 m_2 + 4f_a(B^2 - f_a^2)m_0 m_1 - 2(B^2 - f_a^2)m_0 m_2 - 4f_a m_1 m_2] \quad (3.28)$$

and

$$\epsilon(\hat{B}^2) = \frac{1}{P^2} [(B^4 - 2f_a^2 B^2 + f_a^4)E(m_0^2) + 4f_a^2 E(m_1^2) + E(m_2^2) + 4f_a(B^2 - f_a^2)E(m_0 m_1) - 2(B^2 - f_a^2)E(m_0 m_2) - 4f_a E(m_1 m_2)]. \quad (3.29)$$

Substituting the results of Appendix D for the expected values:

$$\epsilon(\hat{B}^2) = \frac{1}{P^2 T} [(B^4 - 2f_a^2 B^2 + f_a^4) \int_f S_\xi^2(f)df + (4f_a B^2 - 4f_a^3) \int_f fS_\xi^2(f)df + (6f_a^2 - 2B^2) \int_f f^2 S_\xi^2(f)df - 4f_a \int_f f^3 S_\xi^2(f)df + \int_f f^4 S_\xi^2(f)df]. \quad (3.30)$$

This result can be simplified by the same change of variables used before. Performing this substitution and cancelling terms:

$$\epsilon(\hat{B}^2) = \frac{1}{P^2T}\left[\int_f f^4 S_\xi^2(f + f_a)df + B^4 \int_f S_\xi^2(f + f_a)df - 2B^2 \int_f f^2 S_\xi^2(f + f_a)df\right]. \tag{3.31}$$

Or,

$$\epsilon(\hat{B}^2) = \frac{1}{P^2T}\left\{\int_f f^4 S_\xi^2(f + f_a)df + B^2\left[\int_f (B^2 - 2f^2)S_\xi^2(f + f_a)df\right]\right\}. \tag{3.32}$$

The error in this estimator is independent of the true value of f_a. The second term corresponds to the bias of the estimator. A simpler form of this result is

$$\epsilon(\hat{B}^2) = \frac{1}{P^2T}\int_f [f^2 - B^2]^2\, S_\xi^2(f + f_a)df \tag{3.33}$$

which shows the dependence of the expected error on the bandwidth of the spectrum explicitly.