CHAPTER IV

SOME EXAMPLE CALCULATIONS

The value of the results of this investigation can best be demonstrated through some specific examples. The object of these examples is to show the variation of the errors in both the mean frequency estimator and bandwidth estimator as a function of the shape of the spectrum, the true value of the bandwidth B, and the observation time T. Since the rectangular window function is most commonly used in practical situations, all of the results shown here are computed with respect to this function.

It is easiest to conceptualize results that are expressed in the same units as the quantity that is being measured. Hence, the errors computed here are not the mean square errors discussed earlier. Instead, the error in $\hat{f}_a$ that is discussed here is the root mean square, RMS, error $[\epsilon(\hat{f}_a)]^{1/2}$. In the case of the bandwidth estimator, the results are easier to interpret if $\hat{B}$ is discussed instead of $\hat{B}^2$. That is, B is more commonly used as a measure of bandwidth and is, therefore, more familiar.

Let

$$\hat{B}^2 = B^2 + \Delta(B^2). \tag{4.1}$$

Therefore

$$\hat{B} = B[1 + \frac{\Delta(B^2)}{B^2}]^{\frac{1}{2}}. \qquad (4.2)$$

Since for large T

$$\frac{\Delta(B^2)}{B^2} << 1, \qquad (4.3)$$

the square root can be expanded in a power series and approximated by its first three terms,

$$\hat{B} \approx B + \frac{\Delta(B^2)}{2B} - \frac{[\Delta(B^2)]^2}{8B^3}. \qquad (4.4)$$

Therefore

$$E[\hat{B}] \approx B - \frac{\epsilon(\hat{B}^2)}{8B^3}. \qquad (4.5)$$

Since for large T (from Appendix D)

$$E[\hat{B}^2] \approx B^2$$

and from a basic theorem in probability

$$\epsilon(\hat{B}) = E[\hat{B}^2] - \{E[\hat{B}]\}^2, \qquad (4.6)$$

it then follows that

$$\epsilon(\hat{B}) \approx \frac{\epsilon(\hat{B}^2)}{4B^2} - \frac{[\epsilon(\hat{B}^2)]^2}{64\ B^6}. \qquad (4.7)$$

For large T, the second term of this expression is much smaller than the first. Hence, the RMS error in $\hat{B}$ is

$$[\epsilon(\hat{B})]^{\frac{1}{2}} \approx \frac{[\epsilon(\hat{B}^2)]^{\frac{1}{2}}}{2B}. \qquad (4.8)$$

Thus, from the results in Chapter III, it is clear that the two errors to be discussed are

$$[\epsilon(\hat{f}_a)]^{\frac{1}{2}} \approx \frac{1}{P\sqrt{T}} \left[\int_f f^2\ S_\xi^2(f + f_a)df\right]^{\frac{1}{2}} \qquad (4.9)$$

and

$$[\epsilon(\hat{B})]^{\frac{1}{2}} \approx \frac{1}{2PB\sqrt{T}}\left[\int_f [f^2 - B^2]^2 S_\xi^2(f + f_a)df\right]^{\frac{1}{2}} \quad (4.10)$$

<u>Example: A Gaussian Spectrum</u>

Let

$$S(f) = [\sqrt{2\pi}\ B]^{-1}\ e^{-\frac{(f - f_a)^2}{2B^2}}. \qquad (4.11)$$

This is a gaussian shaped spectrum having a total power, P, equal to one, a mean frequency f_a, and a RMS bandwidth B. The expected error for each estimator can now easily be calculated.

For $\hat{f}_a$, the RMS error is

$$[\epsilon(\hat{f}_a)]^{\frac{1}{2}} = T^{-\frac{1}{2}}\left\{[2\pi B^2]^{-1} \int_f f^2\ e^{-\frac{f^2}{B^2}} df\right\}^{\frac{1}{2}}. \quad (4.12)$$

Letting f = BU

$$[\epsilon(\hat{f}_a)]^{\frac{1}{2}} = \left(\frac{B}{2\pi T}\right)^{\frac{1}{2}}\left[\int_u u^2\ e^{-u^2} du\right]^{\frac{1}{2}}. \qquad (4.13)$$

Integrating, the result is

$$[\epsilon(\hat{f}_a)]^{\frac{1}{2}} = \frac{1}{2(\pi)^{\frac{1}{4}}} \left[\frac{B}{T}\right]^{\frac{1}{2}}, \qquad (4.14)$$

Similarly, the error in $\hat{B}$ is

$$[\epsilon(\hat{B})]^{\frac{1}{2}} = [2\sqrt{T}]^{-1}\left\{[2\pi B^2]^{-1} \int_f (f^2 - B^2)^2 e^{-\frac{f^2}{B^2}} df\right\}^{\frac{1}{2}}. \qquad (4.15)$$

Again, let f = BU. Therefore,

$$[\epsilon(\hat{B})]^{\frac{1}{2}} = \frac{B^{\frac{1}{2}}}{2(\pi T)^{\frac{1}{2}}} \left[\int_u (u^2 - 1)^2 e^{-u^2} du\right]^{\frac{1}{2}}. \qquad (4.16)$$

After integration, the result is

$$[\epsilon(\hat{B})]^{\frac{1}{2}} = \frac{(3)^{\frac{1}{2}}}{4(2)^{\frac{1}{2}}(\pi)^{\frac{1}{4}}} \cdot \left[\frac{B}{T}\right]^{\frac{1}{2}} \qquad (4.17)$$

<u>Example: A Rectangular Spectrum</u>

Let

$$S(f) = W^{-1} G_W(f - f_a). \qquad (4.18)$$

This is a rectangular shaped spectrum having a total power equal to one, a mean frequency f_a, and a RMS bandwidth $B = W/(2\sqrt{3})$. Again, the errors for each estimator may be easily calculated.

The error in f_a for this case is

$$[\epsilon(\hat{f}_a)]^{\frac{1}{2}} = \frac{1}{W(T)^{\frac{1}{2}}} \left[\int_{-\frac{W}{2}}^{\frac{W}{2}} f^2 df\right]^{\frac{1}{2}}. \qquad (4.19)$$

Integration yields

$$[\epsilon(\hat{f}_a)]^{\frac{1}{2}} = \frac{1}{2\sqrt{3}} \left(\frac{W}{T}\right)^{\frac{1}{2}}. \qquad (4.20)$$

After substitution of the relationship between W and B, the result becomes

$$[\epsilon(\hat{f}_a)]^{\frac{1}{2}} = \frac{1}{2^{\frac{1}{2}} \; 3^{\frac{1}{4}}} \left(\frac{B}{T}\right)^{\frac{1}{2}}. \qquad (4.21)$$

Similarly, the error in B is

$$[\epsilon(\hat{B})]^{\frac{1}{2}} = \frac{1}{2BW\sqrt{T}} \left[\int_{-\frac{W}{2}}^{\frac{W}{2}} (f^2 - B^2)^2 df\right]^{\frac{1}{2}}. \qquad (4.22)$$

After integration and substitution for W in terms of B, the result is

$$[\epsilon(\hat{B})]^{\frac{1}{2}} = \frac{(3)^{\frac{1}{4}}}{2(5)^{\frac{1}{2}}} \left[\frac{B}{T}\right]^{\frac{1}{2}}. \qquad (4.23)$$

Figure 6 compares the RMS error of the estimates of f_a and B in the two examples. The graph is normalized to a unit observation time, T. Hence, to get a numerical result the ordinate must be divided by the square root of

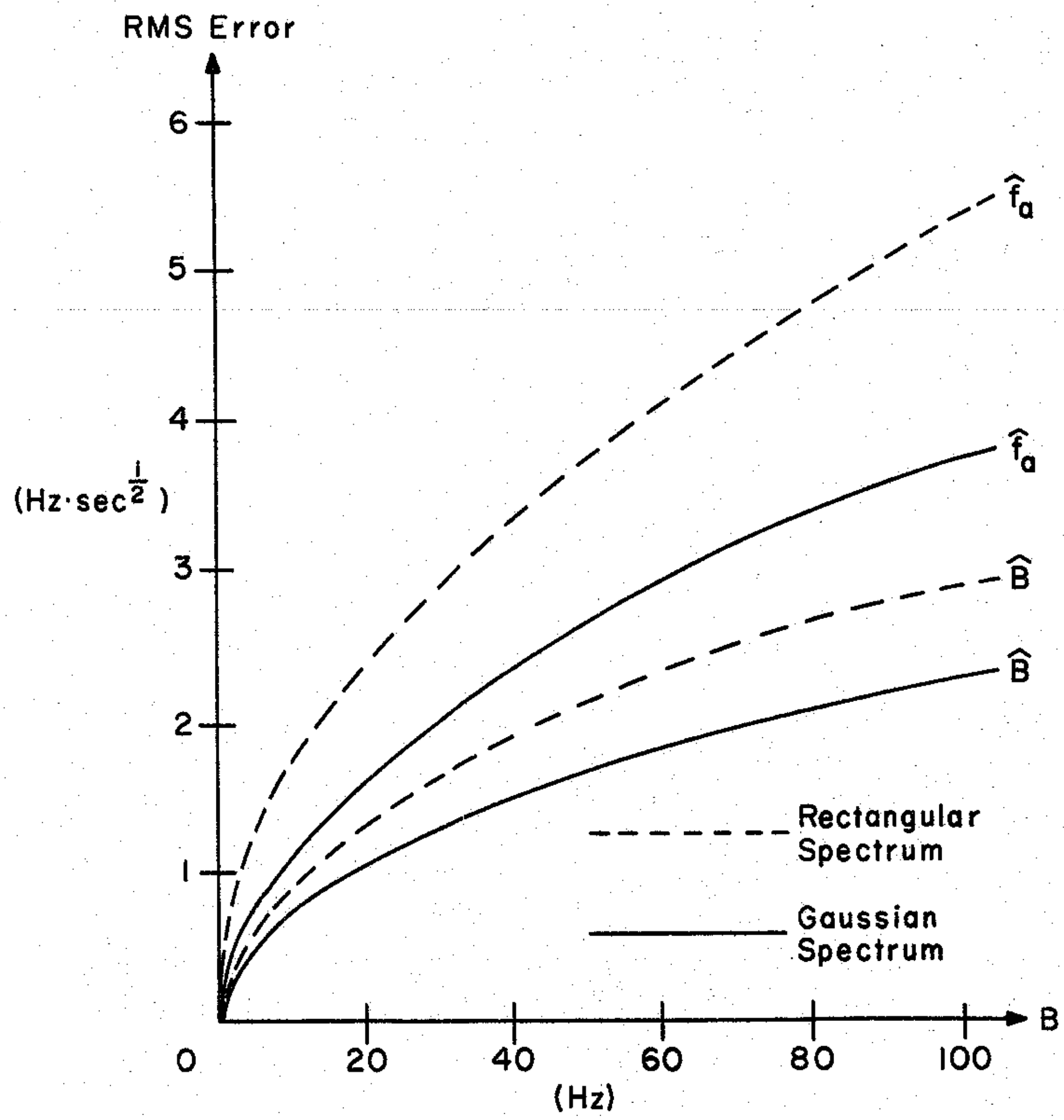

FIG. 6 Normalized RMS Error Versus True Bandwidth

T. Note that in all four cases the relationship is a monotone, increasing function of order $B^{1/2}$. The difference in the errors is a constant multiplicative factor. The error in $\hat{f}_a$ related to the rectangular spectrum is approximately 3.1 db higher than that for the gaussian spectrum, and that in $\hat{B}$ is 2.2 db worse for the rectangular spectrum example than for the gaussian shaped spectrum.

Making the conservative choice of the example with the greater error as the typical case, a statement can be made about the length of time needed to make an accurate measurement. If the true RMS bandwidth of the spectrum is one hundred Hz., the time necessary to make an estimate of f_a, within an average error of one Hz., is approximately 14.1 seconds. If an observation time of one second is allowed, the accuracy is about 3.8 Hz. For the same true bandwidth of one hundred Hz. and a one second observation time, the expected error in $\hat{B}$ is approximately 2.3 Hz. The error in $\hat{B}$ can also be reduced by increasing T. For an error of one Hz., T must be increased to only about 5.3 seconds. Therefore, the estimator for f_a and that for B are both reasonable estimators.