Digital Circuit Design
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Lecture #3

Boolean algebra
Combinational-circuit analysis

Boolean algebra

a.k.a. switching algebra

- deals with Boolean values -- 0, 1

Positive-logic convention

- analog voltages LOW, HIGH --> 0, 1

Negative logic -- seldom used

Signal values denoted by variables (X, Y, FRED, etc.)

Boolean operators

Complement: X′ (opposite of X)
AND: X · Y
OR: X + Y

(binary operators, described functionally by truth table.)

X	Y	X AND Y
0	0	0
0	1	0
1	0	0
1	1	1

X	Y	X OR Y
0	0	0
0	1	1
1	0	1
1	1	1

X	NOT X
0	1
1	0

Huntington Postulates: 1-6 Ch. 2.2
Postulates and Theorems: Table 2-1

More definitions

Literal: a variable or its complement

X, X′, FRED′, CS_L

Expression: literals combined by AND, OR, parentheses, complementation

X+Y

P · Q · R

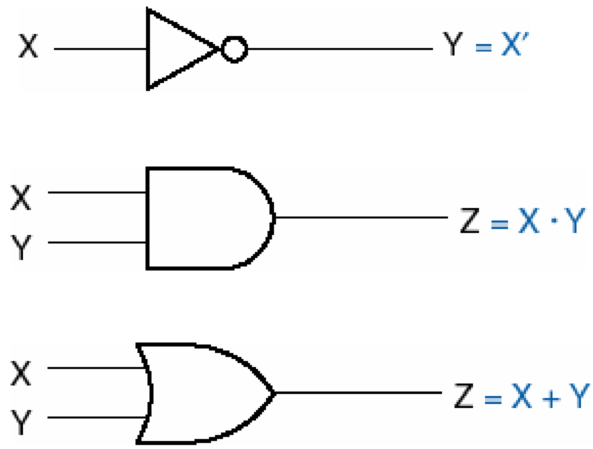
A + B · C

((FRED · Z′) + CS_L · A · B′ · C + Q5) · RESET′

Equation: Variable = expression

P = ((FRED · Z′) + CS_L · A · B′ · C + Q5) · RESET′

Logic symbols



Theorems

(T1)	$X + 0 = X$	(T1′)	$X \cdot 1 = X$	(Identities)
(T2)	$X + 1 = 1$	(T2′)	$X \cdot 0 = 0$	(Null elements)
(T3)	$X + X = X$	(T3′)	$X \cdot X = X$	(Idempotency)
(T4)	$(X')' = X$			(Involution)
(T5)	$X + X' = 1$	(T5′)	$X \cdot X' = 0$	(Complements)

See Table 2-1

Proofs by perfect induction

More Theorems

(T6)	$X + Y = Y + X$	(T6′)	$X \cdot Y = Y \cdot X$	(Commutativity)
(T7)	$(X + Y) + Z = X + (Y + Z)$	(T7′)	$(X \cdot Y) \cdot Z = X \cdot (Y \cdot Z)$	(Associativity)
(T8)	$X \cdot Y + X \cdot Z = X \cdot (Y + Z)$	(T8′)	$(X + Y) \cdot (X + Z) = X + Y \cdot Z$	(Distributivity)
(T9)	$X + X \cdot Y = X$	(T9′)	$X \cdot (X + Y) = X$	(Covering)
(T10)	$X \cdot Y + X \cdot Y' = X$	(T10′)	$(X + Y) \cdot (X + Y') = X$	(Combining)
(T11)	$X \cdot Y + X' \cdot Z + Y \cdot Z = X \cdot Y + X' \cdot Z$			(Consensus)
(T11′)	$(X + Y) \cdot (X' + Z) \cdot (Y + Z) = (X + Y) \cdot (X' + Z)$			

Again Table 2-1

Duality

Swap 0 & 1, AND & OR

Result: Theorems still true

Why?

Each axiom (A1-A5) has a dual (A1′-A5′)

Counterexample:

X + X · Y = X (T9)	X + (X · Y) = X (T9)
X · X + Y = X (dual)	X · (X + Y) = X (dual)
X + Y = X (T3′)	(X · X) + (X · Y) = X (T8)
????????????	X + (X · Y) = X (T3′)
	parentheses,
	operator precedence!

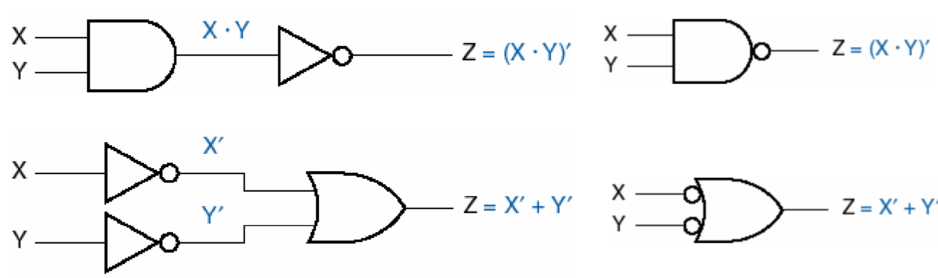
N-variable Theorems

(T12)	$X + X + \cdots + X = X$	(Generalized idempotency)
(T12′)	$X \cdot X \cdot \cdots \cdot X = X$	
(T13)	$(X_1 \cdot X_2 \cdot \cdots \cdot X_n)' = X_1' + X_2' + \cdots + X_n'$	(DeMorgan's theorems)
(T13′)	$(X_1 + X_2 + \cdots + X_n)' = X_1' \cdot X_2' \cdot \cdots \cdot X_n'$	
(T14)	$[F(X_1, X_2, \ldots, X_n, +, \cdot)]' = F(X_1', X_2', \ldots, X_n', \cdot, +)$	(Generalized DeMorgan's theorem)
(T15)	$F(X_1, X_2, \ldots, X_n) = X_1 \cdot F(1, X_2, \ldots, X_n) + X_1' \cdot F(0, X_2, \ldots, X_n)$	(Shannon's expansion theorems)
(T15′)	$F(X_1, X_2, \ldots, X_n) = [X_1 + F(0, X_2, \ldots, X_n)] \cdot [X_1' + F(1, X_2, \ldots, X_n)]$	

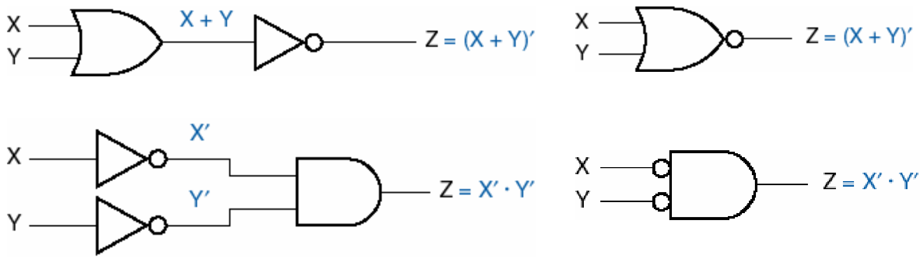
Prove using finite induction

Most important: DeMorgan theorems

DeMorgan Symbol Equivalence

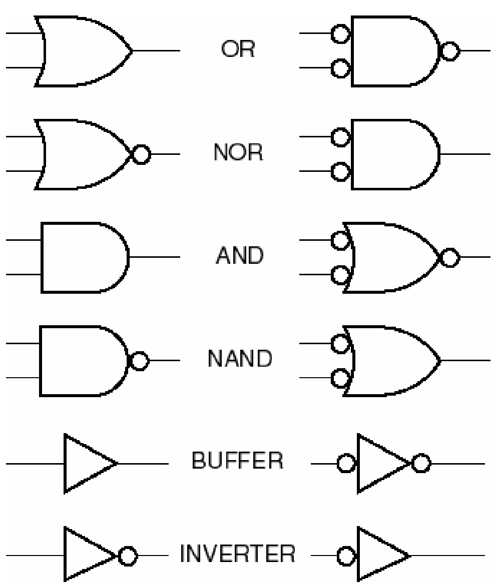


Likewise for OR



(be sure to check errata!)

DeMorgan Symbols



Even more definitions (Sec. 4.1.6)

Product term

Sum-of-products expression

Sum term

Product-of-sums expression

Normal term

Minterm (n variables)

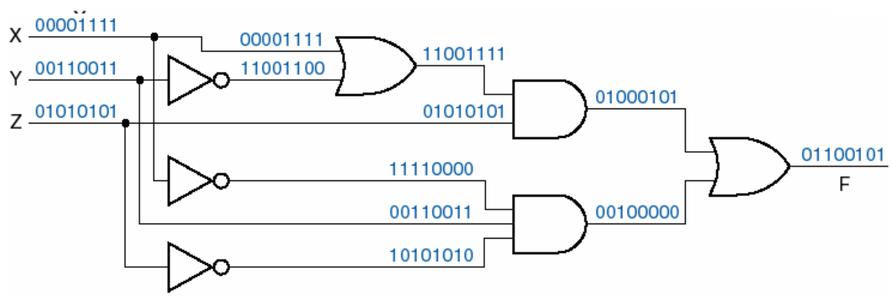
Maxterm (n variables)

Truth table vs. minterms & maxterms

Row	X	Y	Z	F	***Minterm***	***Maxterm***
0	0	0	0	F(0,0,0)	$X' \cdot Y' \cdot Z'$	$X + Y + Z$
1	0	0	1	F(0,0,1)	$X' \cdot Y' \cdot Z$	$X + Y + Z'$
2	0	1	0	F(0,1,0)	$X' \cdot Y \cdot Z'$	$X + Y' + Z$
3	0	1	1	F(0,1,1)	$X' \cdot Y \cdot Z$	$X + Y' + Z'$
4	1	0	0	F(1,0,0)	$X \cdot Y' \cdot Z'$	$X' + Y + Z$
5	1	0	1	F(1,0,1)	$X \cdot Y' \cdot Z$	$X' + Y + Z'$
6	1	1	0	F(1,1,0)	$X \cdot Y \cdot Z'$	$X' + Y' + Z$
7	1	1	1	F(1,1,1)	$X \cdot Y \cdot Z$	$X' + Y' + Z'$

Table 2-3

Combinational analysis



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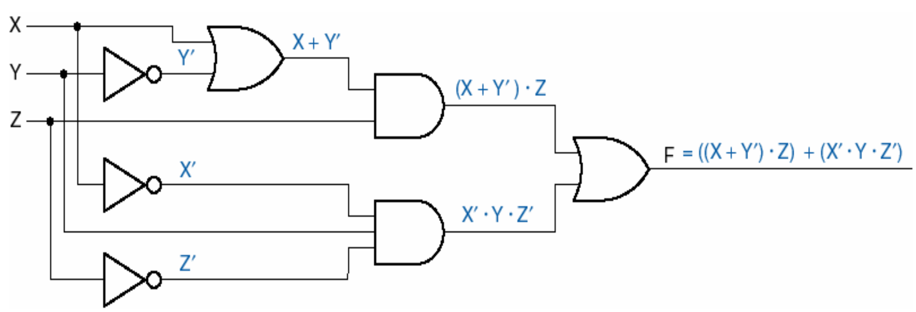
Signal expressions



Multiply out:

$$F = ((X + Y') \cdot Z) + (X' \cdot Y \cdot Z')$$
$$= (X \cdot Z) + (Y' \cdot Z) + (X' \cdot Y \cdot Z')$$

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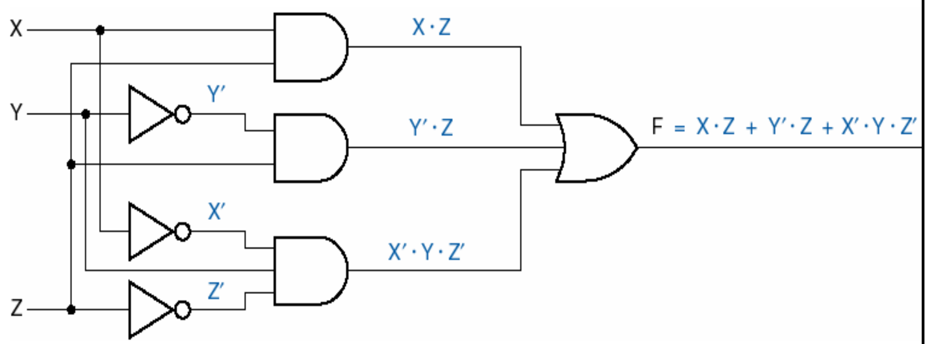
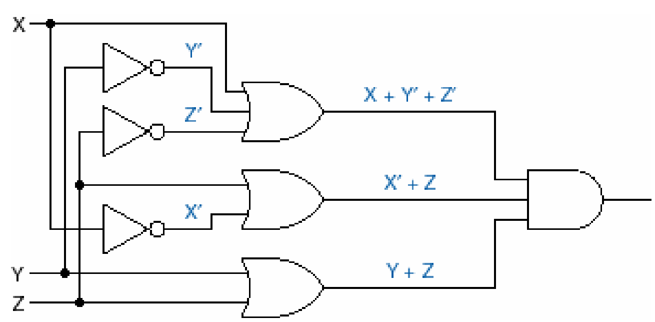
New circuit, same function



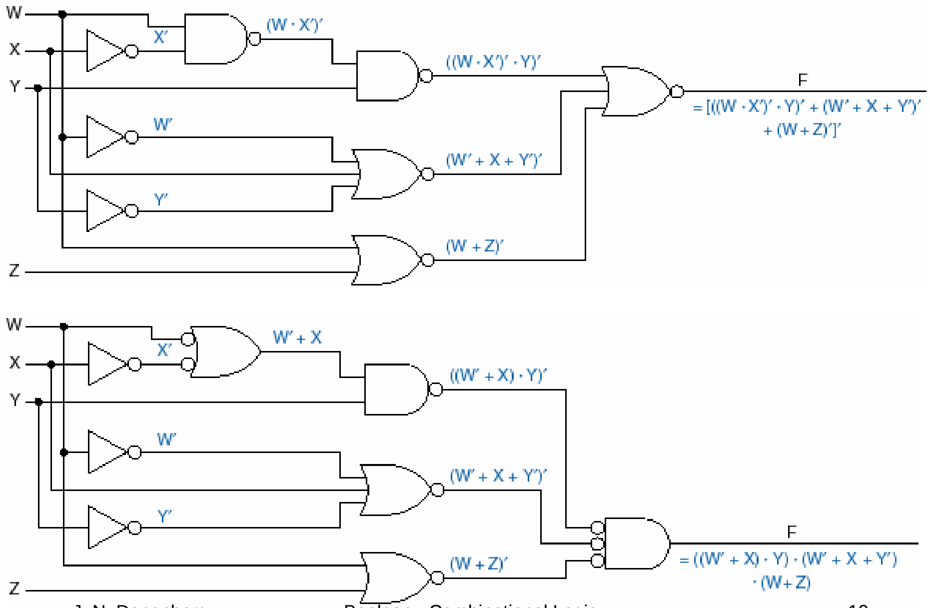
Add out logic function

$$\begin{aligned} F &= ((X+Y')\cdot Z)+(X'\cdot Y\cdot Z') \\ &= (X+Y'+X')\cdot(X+Y'+Y)\cdot(X+Y'+Z')\cdot(Z+X')\cdot(Z+Y)\cdot(Z+Z') \\ &= 1\cdot 1\cdot(X+Y'+Z')\cdot(X'+Z)\cdot(Y+Z)\cdot 1 \\ &= (X+Y'+Z')\cdot(X'+Z)\cdot(Y+Z) \end{aligned}$$

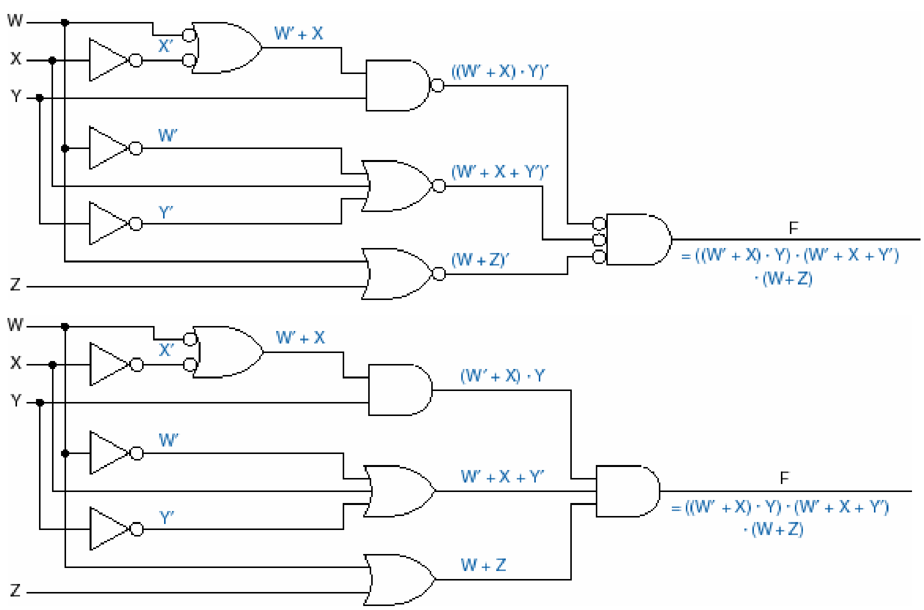
Circuit:



Shortcut: Symbol substitution



Different circuit, same function



Another example

G(W, X, Y, Z) = W · X ·Y + Y · Z