

Multirate Digital Signal Processing

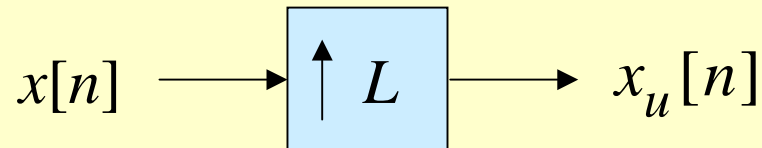
Basic Sampling Rate Alteration Devices

- **Up-sampler** - Used to increase the sampling rate by an integer factor
- **Down-sampler** - Used to decrease the sampling rate by an integer factor

Up-Sampler

Time-Domain Characterization

- An up-sampler with an **up-sampling factor** L , where L is a positive integer, develops an output sequence $x_u[n]$ with a sampling rate that is L times larger than that of the input sequence $x[n]$
- **Block-diagram representation**



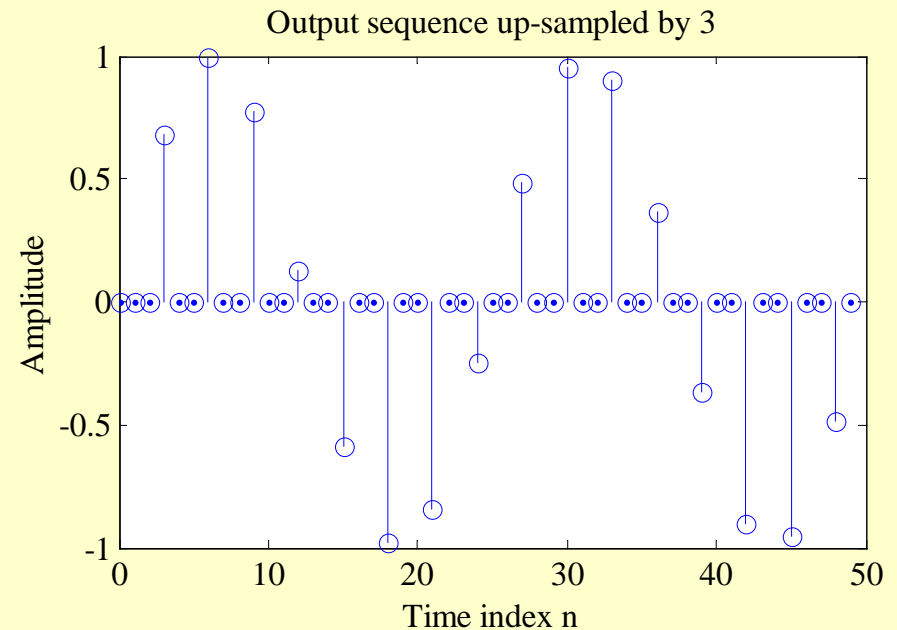
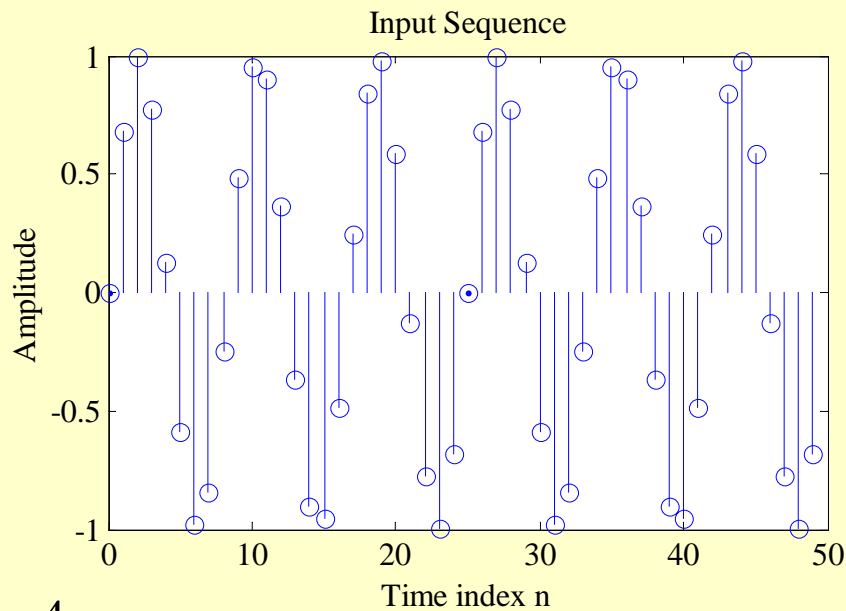
Up-Sampler

- Up-sampling operation is implemented by inserting $L-1$ equidistant zero-valued samples between two consecutive samples of $x[n]$
- Input-output relation

$$x_u[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \dots \\ 0, & \text{otherwise} \end{cases}$$

Up-Sampler

- Figure below shows the up-sampling by a factor of 3 of a sinusoidal sequence with a frequency of 0.12 Hz obtained using Program 10_1



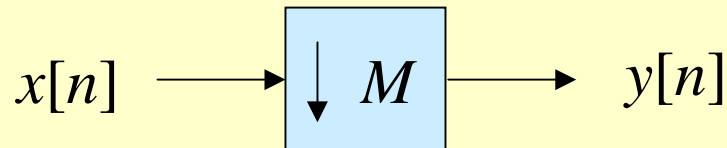
Up-Sampler

- In practice, the zero-valued samples inserted by the up-sampler are replaced with appropriate nonzero values using some type of filtering process
- Process is called **interpolation** and will be discussed later

Down-Sampler

Time-Domain Characterization

- An down-sampler with a **down-sampling factor** M , where M is a positive integer, develops an output sequence $y[n]$ with a sampling rate that is $(1/M)$ -th of that of the input sequence $x[n]$
- **Block-diagram representation**



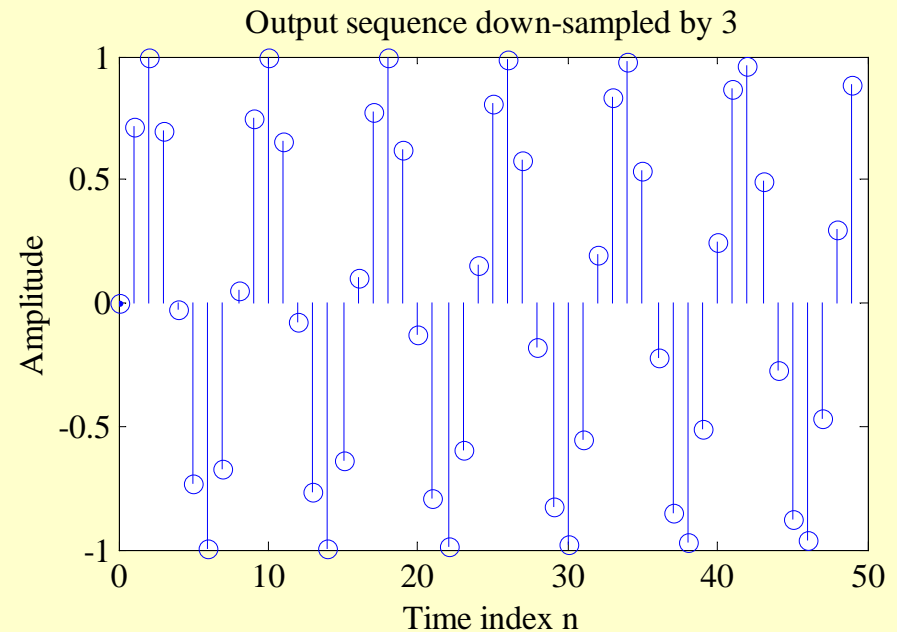
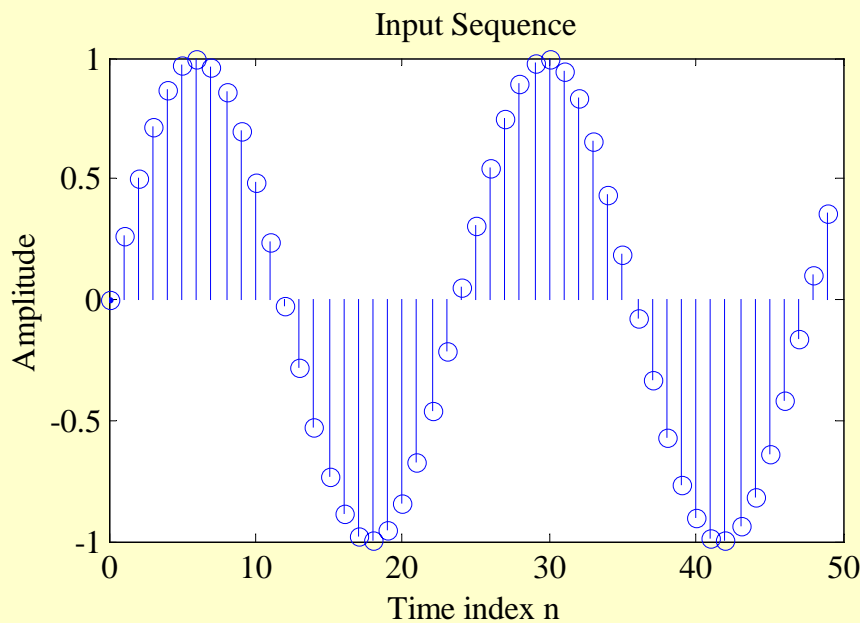
Down-Sampler

- Down-sampling operation is implemented by keeping every M -th sample of $x[n]$ and removing $M - 1$ in-between samples to generate $y[n]$
- Input-output relation

$$y[n] = x[nM]$$

Down-Sampler

- Figure below shows the down-sampling by a factor of 3 of a sinusoidal sequence of frequency 0.042 Hz obtained using Program 10_2

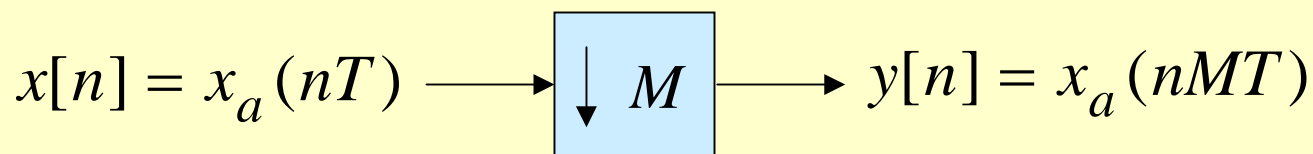


Basic Sampling Rate Alteration Devices

- Sampling periods have not been explicitly shown in the block-diagram representations of the up-sampler and the down-sampler
- This is for simplicity and the fact that the mathematical theory of multirate systems can be understood without bringing the sampling period T or the sampling frequency F_T into the picture

Down-Sampler

- Figure below shows explicitly the time-dimensions for the down-sampler



Input sampling frequency

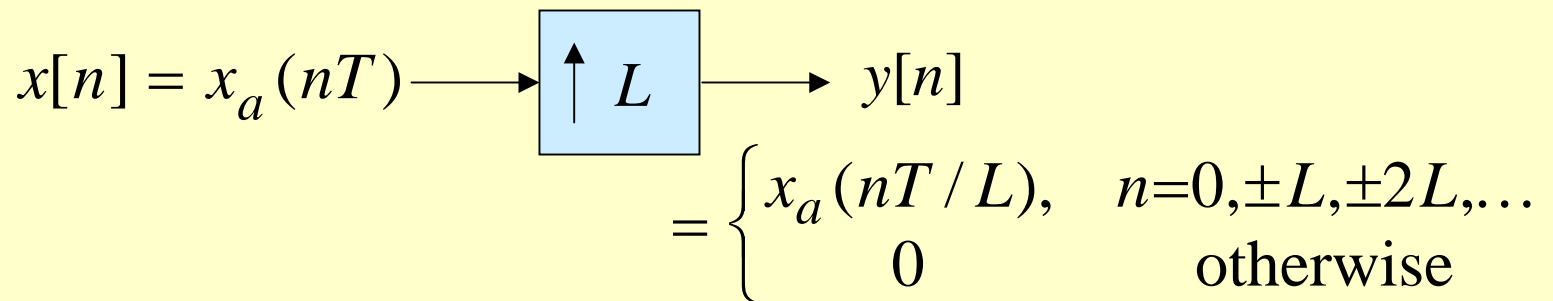
$$F_T = \frac{1}{T}$$

Output sampling frequency

$$F_T' = \frac{F_T}{M} = \frac{1}{T'}$$

Up-Sampler

- Figure below shows explicitly the time-dimensions for the up-sampler



Input sampling frequency

$$F_T = \frac{1}{T}$$

Output sampling frequency

$$F_T' = LF_T = \frac{1}{T}$$

Basic Sampling Rate Alteration Devices

- The up-sampler and the down-sampler are linear but time-varying discrete-time systems
- We illustrate the time-varying property of a down-sampler
- The time-varying property of an up-sampler can be proved in a similar manner

Basic Sampling Rate Alteration Devices

- Consider a factor-of- M down-sampler defined by $y[n] = x[nM]$
- Its output $y_1[n]$ for an input $x_1[n] = x[n - n_0]$ is then given by

$$y_1[n] = x_1[Mn] = x[Mn - n_0]$$

- From the input-output relation of the down-sampler we obtain

$$\begin{aligned} y[n - n_0] &= x[M(n - n_0)] \\ &= x[Mn - Mn_0] \neq y_1[n] \end{aligned}$$

Up-Sampler

Frequency-Domain Characterization

- Consider first a factor-of-2 up-sampler whose input-output relation in the time-domain is given by

$$x_u[n] = \begin{cases} x[n/2], & n = 0, \pm 2, \pm 4, \dots \\ 0, & \text{otherwise} \end{cases}$$

Up-Sampler

- In terms of the z -transform, the input-output relation is then given by

$$X_u(z) = \sum_{n=-\infty}^{\infty} x_u[n] z^{-n} = \sum_{\substack{n=-\infty \\ n \text{ even}}}^{\infty} x[n/2] z^{-n}$$

$$= \sum_{m=-\infty}^{\infty} x[2m] z^{-n} = X(z^2)$$

Up-Sampler

- In terms of the z -transform, the input-output relation is then given by

$$\begin{aligned} X_u(z) &= \sum_{n=-\infty}^{\infty} x_u[n] z^{-n} = \sum_{\substack{n=-\infty \\ n \text{ even}}}^{\infty} x[n/2] z^{-n} \\ &= \sum_{m=-\infty}^{\infty} x[2m] z^{-n} = X(z^2) \end{aligned}$$

Up-Sampler

- In a similar manner, we can show that for a factor-of- L up-sampler

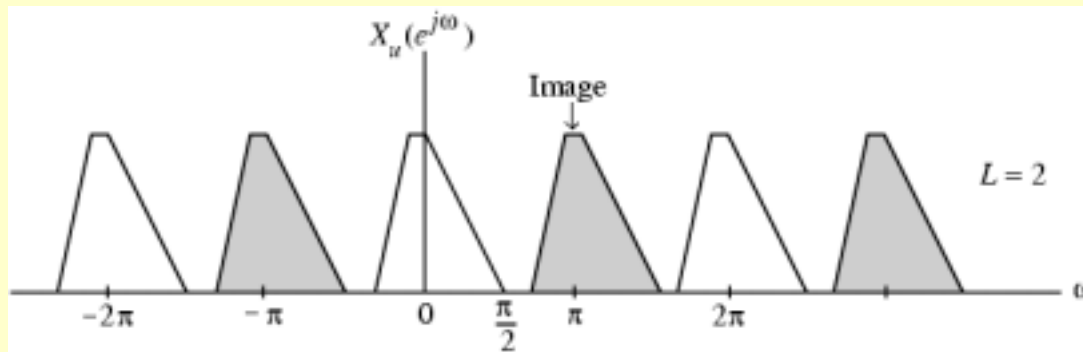
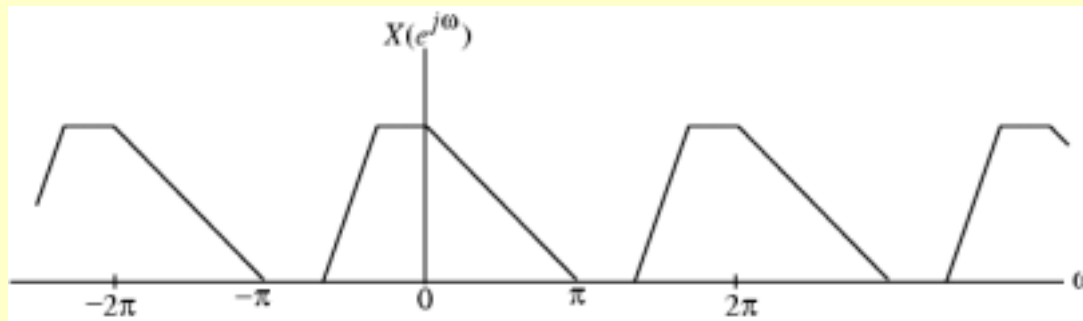
$$X_u(z) = X(z^L)$$

- On the unit circle, for $z = e^{j\omega}$, the input-output relation is given by

$$X_u(e^{j\omega}) = X(e^{j\omega L})$$

Up-Sampler

- Figure below shows the relation between $X(e^{j\omega})$ and $X_u(e^{j\omega})$ for $L = 2$ in the case of a typical sequence $x[n]$



Up-Sampler

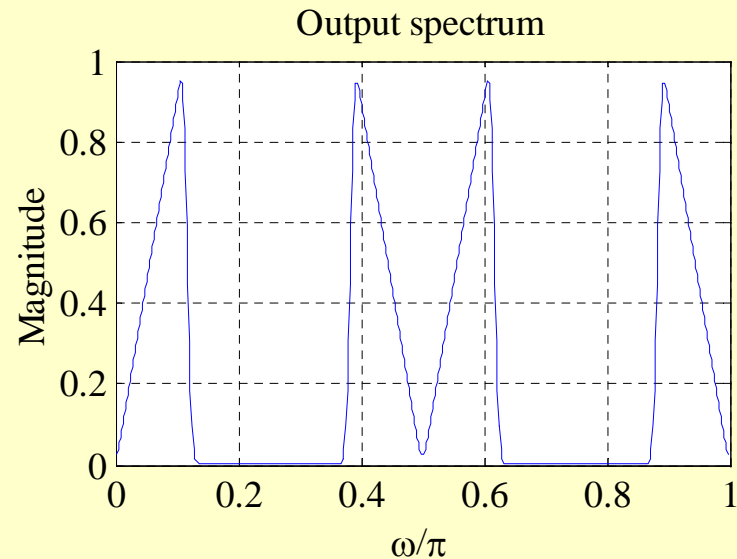
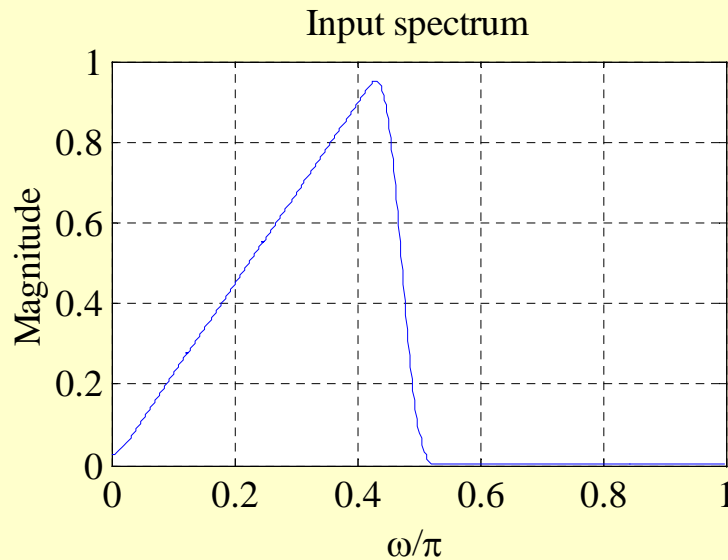
- As can be seen, a factor-of-2 sampling rate expansion leads to a compression of $X(e^{j\omega})$ by a factor of 2 and a 2-fold repetition in the baseband $[0, 2\pi]$
- This process is called **imaging** as we get an additional “**image**” of the input spectrum

Up-Sampler

- Similarly in the case of a factor-of- L sampling rate expansion, there will be $L-1$ additional images of the input spectrum in the baseband
- Lowpass filtering of $x_u[n]$ removes the $L-1$ images and in effect “fills in” the zero-valued samples in $x_u[n]$ with interpolated sample values

Up-Sampler

- Program 10_3 can be used to illustrate the frequency-domain properties of the up-sampler shown below for $L = 4$



Down-Sampler

Frequency-Domain Characterization

- Applying the z -transform to the input-output relation of a factor-of- M down-sampler

$$y[n] = x[Mn]$$

we get

$$Y(z) = \sum_{n=-\infty}^{\infty} x[Mn] z^{-n}$$

- The expression on the right-hand side cannot be directly expressed in terms of $X(z)$

Down-Sampler

- To get around this problem, define a new sequence $x_{\text{int}}[n]$:

$$x_{\text{int}}[n] = \begin{cases} x[n], & n = 0, \pm M, \pm 2M, \dots \\ 0, & \text{otherwise} \end{cases}$$

- Then

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} x[Mn] z^{-n} = \sum_{n=-\infty}^{\infty} x_{\text{int}}[Mn] z^{-n} \\ &= \sum_{k=-\infty}^{\infty} x_{\text{int}}[k] z^{-k/M} = X_{\text{int}}(z^{1/M}) \end{aligned}$$

Down-Sampler

- Now, $x_{\text{int}}[n]$ can be formally related to $x[n]$ through

$$x_{\text{int}}[n] = c[n] \cdot x[n]$$

where

$$c[n] = \begin{cases} 1, & n = 0, \pm M, \pm 2M, \dots \\ 0, & \text{otherwise} \end{cases}$$

- A convenient representation of $c[n]$ is given by

$$c[n] = \frac{1}{M} \sum_{k=0}^{M-1} W_M^{kn}$$

where $W_M = e^{-j2\pi/M}$

Down-Sampler

- Taking the z -transform of $x_{\text{int}}[n] = c[n] \cdot x[n]$ and making use of

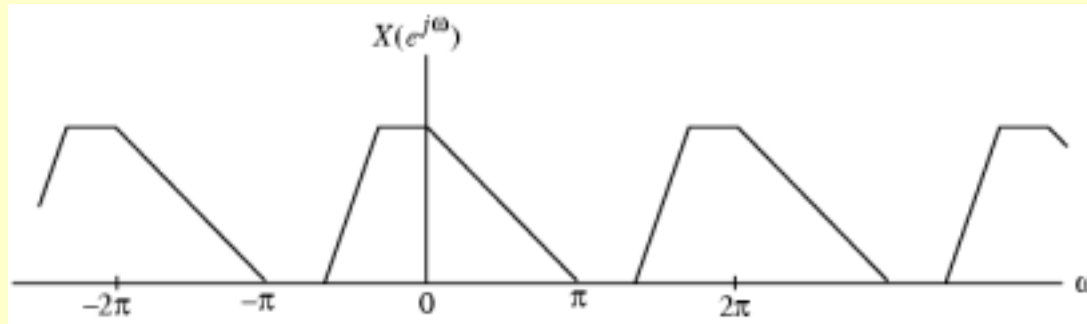
$$c[n] = \frac{1}{M} \sum_{k=0}^{M-1} W_M^{kn}$$

we arrive at

$$\begin{aligned} X_{\text{int}}(z) &= \sum_{n=-\infty}^{\infty} c[n]x[n]z^{-n} = \frac{1}{M} \sum_{n=-\infty}^{\infty} \left(\sum_{k=0}^{M-1} W_M^{kn} \right) x[n]z^{-n} \\ &= \frac{1}{M} \sum_{k=0}^{M-1} \left(\sum_{n=-\infty}^{\infty} x[n]W_M^{kn} z^{-n} \right) = \frac{1}{M} \sum_{k=0}^{M-1} X(zW_M^{-k}) \end{aligned}$$

Down-Sampler

- Consider a factor-of-2 down-sampler with an input $x[n]$ whose spectrum is as shown below

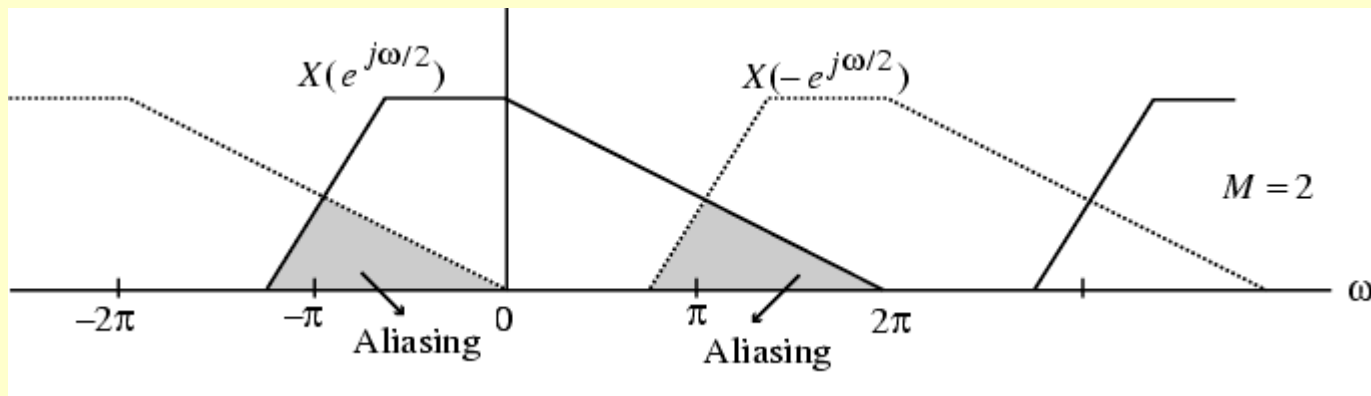


- The DTFTs of the output and the input sequences of this down-sampler are then related as

$$Y(e^{j\omega}) = \frac{1}{2} \{ X(e^{j\omega/2}) + X(-e^{j\omega/2}) \}$$

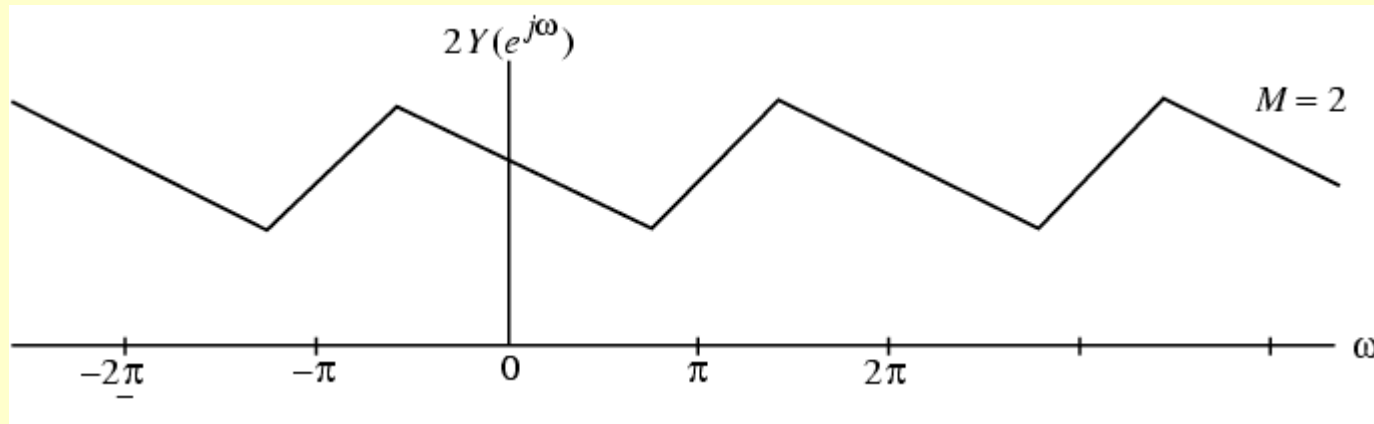
Down-Sampler

- Now $X(-e^{j\omega/2}) = X(e^{j(\omega-2\pi)/2})$ implying that the second term $X(-e^{j\omega/2})$ in the previous equation is simply obtained by shifting the first term $X(e^{j\omega/2})$ to the right by an amount 2π as shown below



Down-Sampler

- The plots of the two terms have an overlap, and hence, in general, the original “shape” of $X(e^{j\omega})$ is lost when $x[n]$ is down-sampled as indicated below



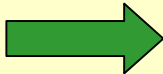
Down-Sampler

- This overlap causes the **aliasing** that takes place due to under-sampling
- There is no overlap, i.e., no aliasing, only if
$$X(e^{j\omega}) = 0 \quad \text{for } |\omega| \geq \pi/2$$
- **Note:** $Y(e^{j\omega})$ is indeed periodic with a period 2π , even though the stretched version of $X(e^{j\omega})$ is periodic with a period 4π

Down-Sampler

- For the general case, the relation between the DTFTs of the output and the input of a factor-of- M down-sampler is given by

$$Y(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X(e^{j(\omega-2\pi k)/M})$$

-  $Y(e^{j\omega})$ is a sum of M uniformly shifted and stretched versions of $X(e^{j\omega})$ and scaled by a factor of $1/M$

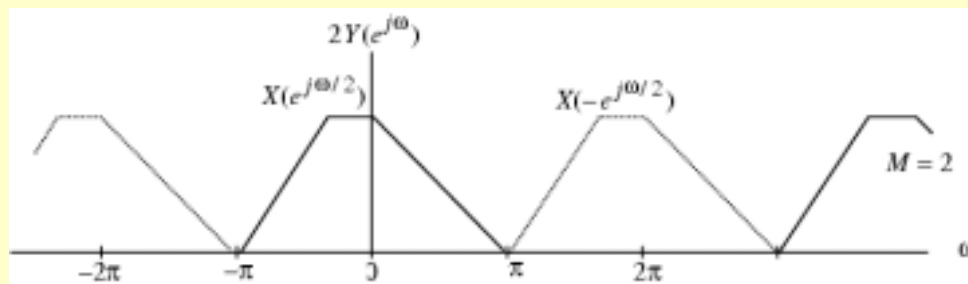
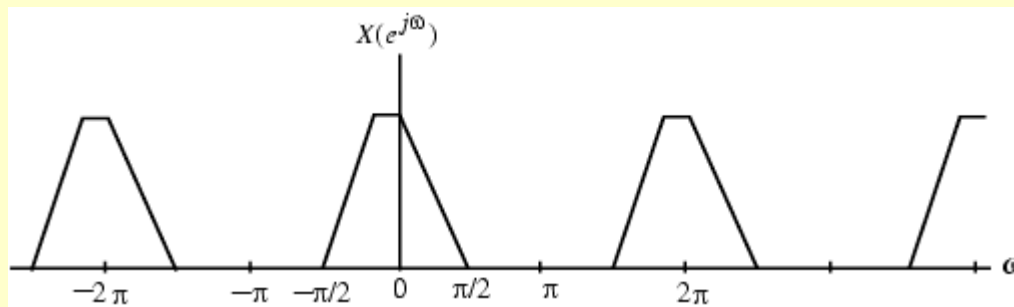
Down-Sampler

- Aliasing is absent if and only if

$$X(e^{j\omega}) = 0 \text{ for } |\omega| \geq \pi / M$$

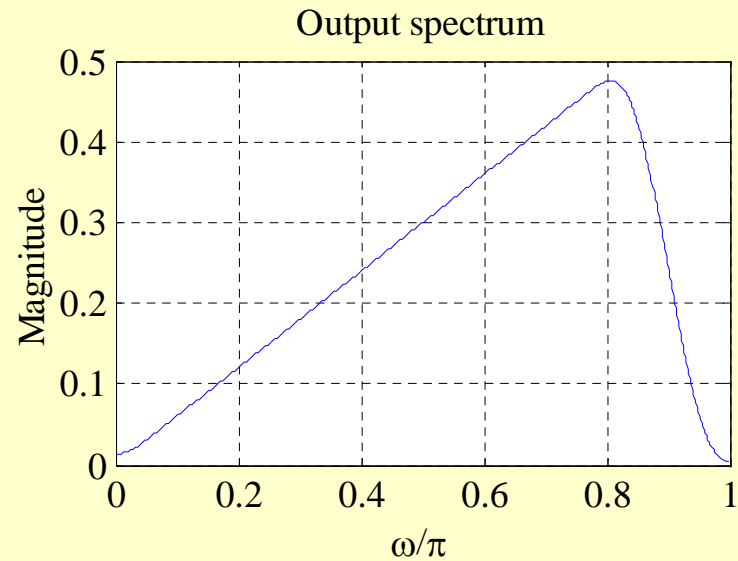
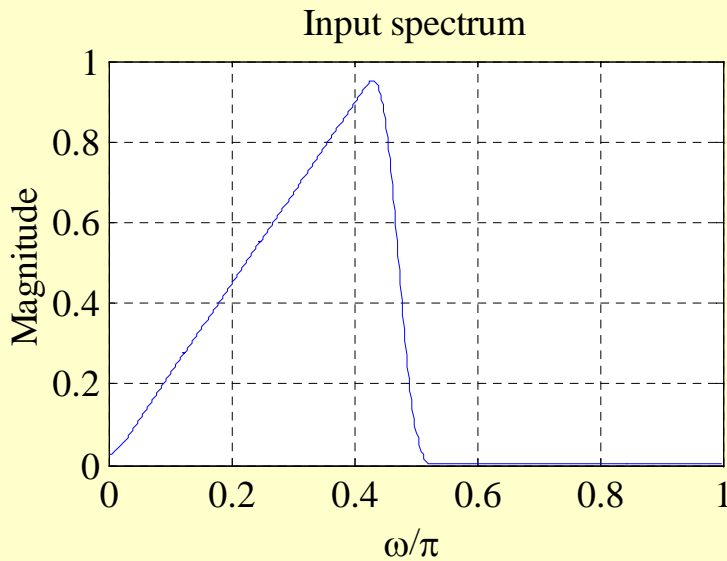
as shown below for $M = 2$

$$X(e^{j\omega}) = 0 \text{ for } |\omega| \geq \pi / 2$$



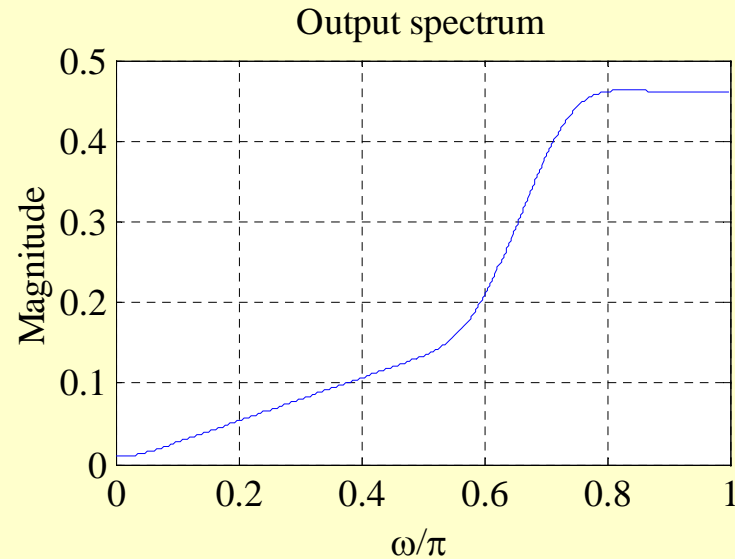
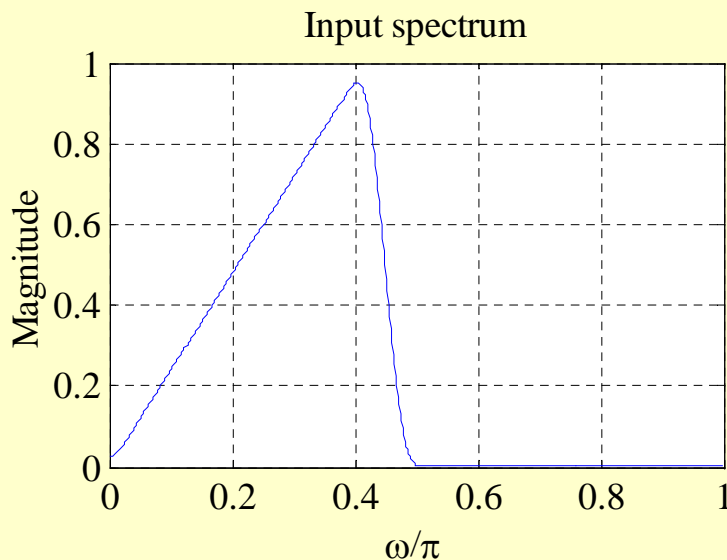
Down-Sampler

- Program 10_4 can be used to illustrate the frequency-domain properties of the up-sampler shown below for $M = 2$



Down-Sampler

- The input and output spectra of a down-sampler with $M = 3$ obtained using Program 10-4 are shown below



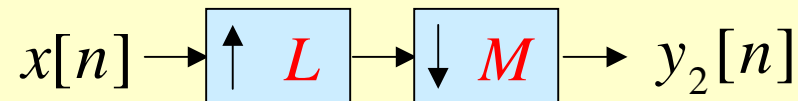
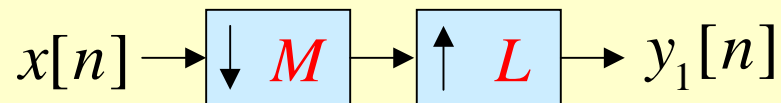
- Effect of aliasing can be clearly seen

Cascade Equivalences

- A complex multirate system is formed by an interconnection of the up-sampler, the down-sampler, and the components of an LTI digital filter
- In many applications these devices appear in a cascade form
- An interchange of the positions of the branches in a cascade often can lead to a computationally efficient realization

Cascade Equivalences

- To implement a fractional change in the sampling rate we need to employ a cascade of an up-sampler and a down-sampler
- Consider the two cascade connections shown below



Cascade Equivalences

- A cascade of a factor-of- M down-sampler and a factor-of- L up-sampler is interchangeable with no change in the input-output relation:

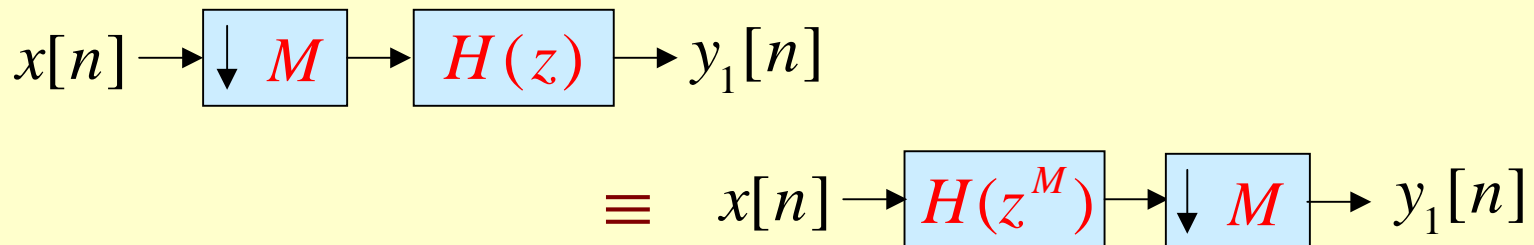
$$y_1[n] = y_2[n]$$

if and only if M and L are relatively prime, i.e., M and L do not have any common factor that is an integer $k > 1$

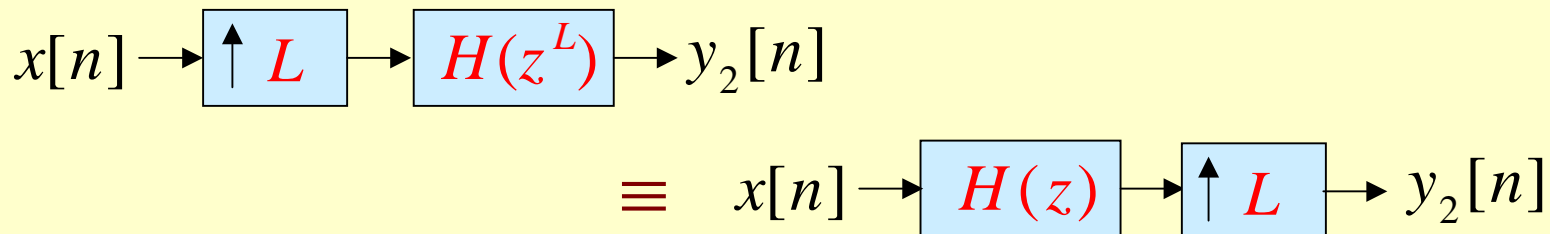
Cascade Equivalences

- Two other cascade equivalences are shown below

Cascade equivalence #1



Cascade equivalence #2



Filters in Sampling Rate Alteration Systems

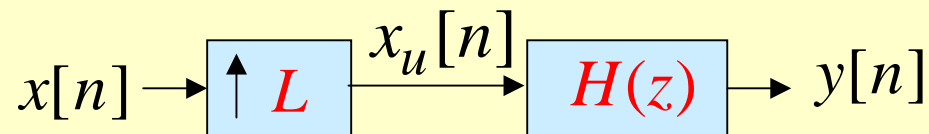
- From the sampling theorem it is known that a the sampling rate of a critically sampled discrete-time signal with a spectrum occupying the full Nyquist range cannot be reduced any further since such a reduction will introduce aliasing
- Hence, the bandwidth of a critically sampled signal must be reduced by lowpass filtering before its sampling rate is reduced by a down-sampler

Filters in Sampling Rate Alteration Systems

- Likewise, the zero-valued samples introduced by an up-sampler must be interpolated to more appropriate values for an effective sampling rate increase
- We shall show next that this interpolation can be achieved simply by digital lowpass filtering
- We now develop the frequency response specifications of these lowpass filters

Filter Specifications

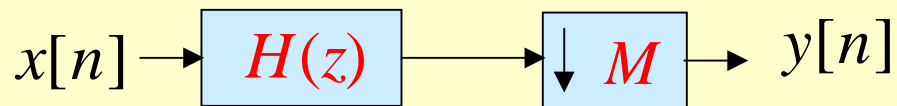
- Since up-sampling causes periodic repetition of the basic spectrum, the unwanted images in the spectra of the up-sampled signal $x_u[n]$ must be removed by using a lowpass filter $H(z)$, called the **interpolation filter**, as indicated below



- The above system is called an **interpolator**

Filter Specifications

- On the other hand, prior to down-sampling, the signal $v[n]$ should be bandlimited to $|\omega| < \pi / M$ by means of a lowpass filter, called the **decimation filter**, as indicated below to avoid aliasing caused by down-sampling



- The above system is called a **decimator**

Interpolation Filter Specifications

- Assume $x[n]$ has been obtained by sampling a continuous-time signal $x_a(t)$ at the Nyquist rate
- If $X_a(j\Omega)$ and $X(e^{j\omega})$ denote the Fourier transforms of $x_a(t)$ and $x[n]$, respectively, then it can be shown

$$X(e^{j\omega}) = \frac{1}{T_0} \sum_{k=-\infty}^{\infty} X_a\left(\frac{j\omega - j2\pi k}{T_0}\right)$$

- where T_0 is the sampling period

Interpolation Filter Specifications

- Since the sampling is being performed at the Nyquist rate, there is no overlap between the shifted spectras of $X(j\omega/T_0)$
- If we instead sample $x_a(t)$ at a much higher rate $T = L \cdot T_0$ yielding $y[n]$, its Fourier transform $Y(e^{j\omega})$ is related to $X_a(j\Omega)$ through

$$Y(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a\left(\frac{j\omega - j2\pi k}{T}\right) = \frac{L}{T_0} \sum_{k=-\infty}^{\infty} X_a\left(\frac{j\omega - j2\pi k}{T_0/L}\right)$$

Interpolation Filter Specifications

- On the other hand, if we pass $x[n]$ through a factor-of- L up-sampler generating $x_u[n]$, the relation between the Fourier transforms of $x[n]$ and $x_u[n]$ are given by

$$X_u(e^{j\omega}) = X(e^{j\omega L})$$

- It therefore follows that if $x_u[n]$ is passed through an ideal lowpass filter $H(z)$ with a cutoff at π/L and a gain of L , the output of the filter will be precisely $y[n]$

Interpolation Filter Specifications

- In practice, a transition band is provided to ensure the realizability and stability of the lowpass interpolation filter $H(z)$
- Hence, the desired lowpass filter should have a stopband edge at $\omega_s = \pi / L$ and a passband edge ω_p close to ω_s to reduce the distortion of the spectrum of $x[n]$

Interpolation Filter Specifications

- If ω_c is the highest frequency that needs to be preserved in $x[n]$, then

$$\omega_p = \omega_c / L$$

- Summarizing the specifications of the lowpass interpolation filter are thus given by

$$|H(e^{j\omega})| = \begin{cases} L, & |\omega| \leq \omega_c / L \\ 0, & \pi / L \leq |\omega| \leq \pi \end{cases}$$

Decimation Filter Specifications

- In a similar manner, we can develop the specifications for the lowpass decimation filter that are given by

$$\left| H(e^{j\omega}) \right| = \begin{cases} 1, & |\omega| \leq \omega_c / M \\ 0, & \pi / M \leq |\omega| \leq \pi \end{cases}$$

Filter Design Methods

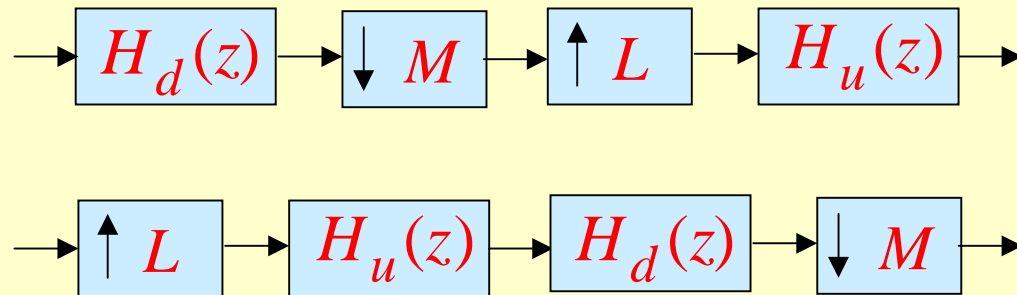
- The design of the filter $H(z)$ is a standard IIR or FIR lowpass filter design problem
- Any one of the techniques outlined in Chapter 7 can be applied for the design of these lowpass filters

Filters for Fractional Sampling Rate Alteration

- A fractional change in the sampling rate can be achieved by cascading a factor-of- M decimator with a factor-of- L interpolator, where M and L are positive integers
- Such a cascade is equivalent to a decimator with a decimation factor of M/L or an interpolator with an interpolation factor of L/M

Filters for Fractional Sampling Rate Alteration

- There are two possible such cascade connections as indicated below

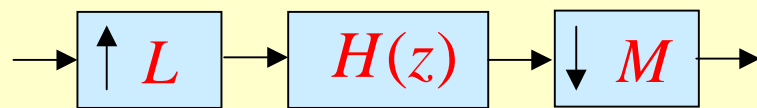


- The second scheme is more computationally efficient since only one of the filters, $H_u(z)$ or $H_d(z)$, is adequate to serve as both the interpolation and the decimation filter

Filters for Fractional Sampling Rate Alteration

- Hence, the desired configuration for the fractional sampling rate alteration is as indicated below where the lowpass filter $H(z)$ has a stopband edge frequency given by

$$\omega_s = \min\left(\frac{\pi}{L}, \frac{\pi}{M}\right)$$

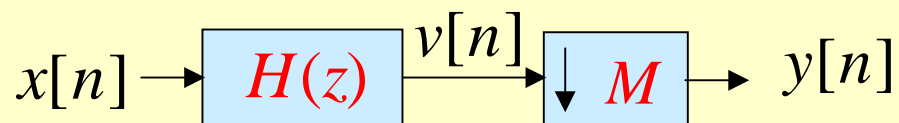


Computational Requirements

- The lowpass decimation or interpolation filter can be designed either as an FIR or an IIR digital filter
- In the case of single-rate digital signal processing, IIR digital filters are, in general, computationally more efficient than equivalent FIR digital filters, and are therefore preferred where computational cost needs to be minimized

Computational Requirements

- This issue is not quite the same in the case of multirate digital signal processing
- To illustrate this point further, consider the factor-of- M decimator shown below



- If the decimation filter $H(z)$ is an FIR filter of length N implemented in a direct form, then

$$v[n] = \sum_{m=0}^{N-1} h[m] x[n-m]$$

Computational Requirements

- Now, the down-sampler keeps only every M -th sample of $v[n]$ at its output
- Hence, it is sufficient to compute $v[n]$ only for values of n that are multiples of M and skip the computations of in-between samples
- This leads to a factor of M savings in the computational complexity

Computational Requirements

- Now assume $H(z)$ to be an IIR filter of order K with a transfer function

$$\frac{V(z)}{X(z)} = H(z) = \frac{P(z)}{D(z)}$$

where

$$P(z) = \sum_{n=0}^K p_n z^{-n}$$

$$D(z) = 1 + \sum_{n=1}^K d_n z^{-n}$$

Computational Requirements

- Its direct form implementation is given by

$$w[n] = -d_1 w[n-1] - d_2 w[n-2] - \dots$$

$$- d_K w[n-K] + x[n]$$

$$v[n] = p_0 w[n] + p_1 w[n-1] + \dots + p_K w[n-K]$$

- Since $v[n]$ is being down-sampled, it is sufficient to compute $v[n]$ only for values of n that are integer multiples of M

Computational Requirements

- However, the intermediate signal $w[n]$ must be computed for all values of n
- For example, in the computation of
$$v[M] = p_0 w[M] + p_1 w[M-1] + \dots + p_K w[M-K]$$
 $K+1$ successive values of $w[n]$ are still required
- As a result, the savings in the computation in this case is going to be less than a factor of M

Computational Requirements

- Example - We compare the computational complexity of various implementations of a factor-of- M decimator
- Let the sampling frequency be F_T
- Then the number of multiplications per second, to be denoted as \mathcal{R}_M , are as follows for various computational schemes

Computational Requirements

- **FIR** $H(z)$ of length N :

$$\mathcal{R}_{M, FIR} = N \times F_T$$

- **FIR** $H(z)$ of length N followed by a down-sampler:

$$\mathcal{R}_{M, FIR-DEC} = N \times F_T / M$$

- **IIR** $H(z)$ of order K :

$$\mathcal{R}_{M, IIR} = (2K + 1) \times F_T$$

- **IIR** $H(z)$ of order K followed by a down-sampler :

$$59 \quad \mathcal{R}_{M, IIR-DEC} = K \times F_T + (K + 1) \times F_T / M$$

Computational Requirements

- In the FIR case, savings in computations is by a factor of M
- In the IIR case, savings in computations is by a factor of $M(2K+1)/[(M+1)K+1]$, which is not significant for large K
- For $M = 10$ and $K = 9$, the savings is only by a factor of 1.9
- There are certain cases where the IIR filter can be computationally more efficient

Computational Requirements

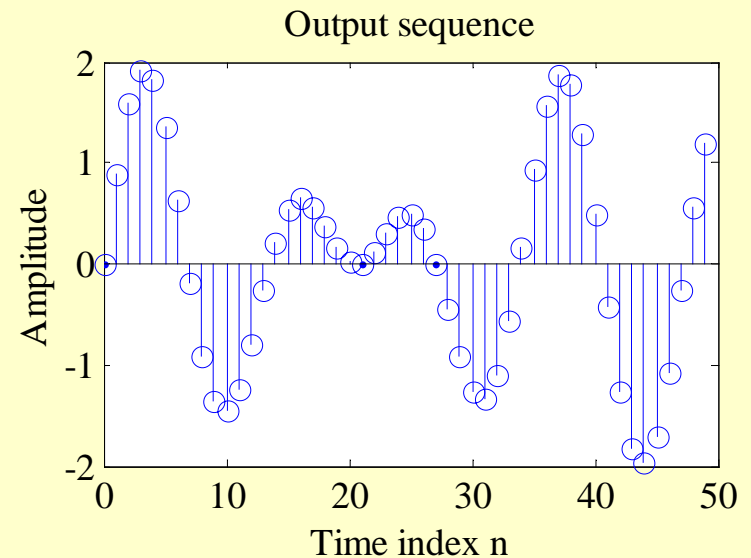
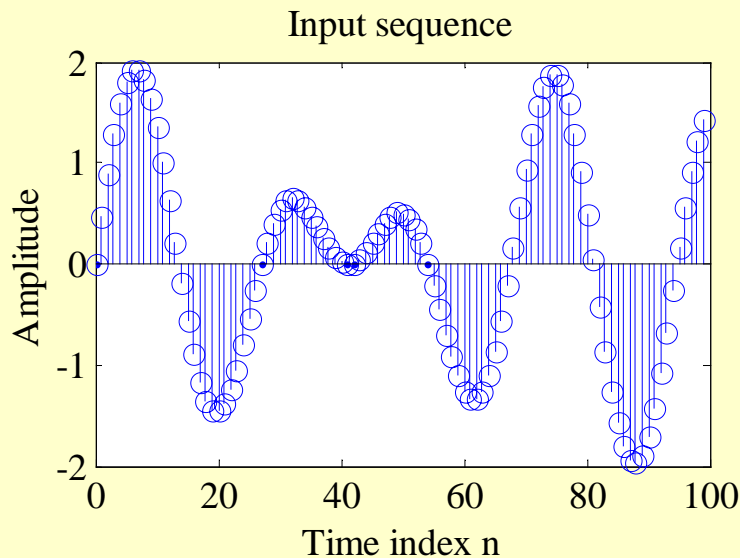
- For the case of interpolator design, very similar arguments hold
- If $H(z)$ is an FIR interpolation filter, then the computational savings is by a factor of L (since $v[n]$ has $L-1$ zeros between its consecutive nonzero samples)
- On the other hand, computational savings is significantly less with IIR filters

Sampling Rate Alteration Using MATLAB

- The function `decimate` can be employed to reduce the sampling rate of an input signal vector \mathbf{x} by an integer factor M to generate the output signal vector \mathbf{y}
- The decimation of a sequence by a factor of M can be obtained using [Program 10_5](#) which employs the function `decimate`

Sampling Rate Alteration Using MATLAB

- Example - The input and output plots of a factor-of-2 decimator designed using the Program 10_5 are shown below



Sampling Rate Alteration Using MATLAB

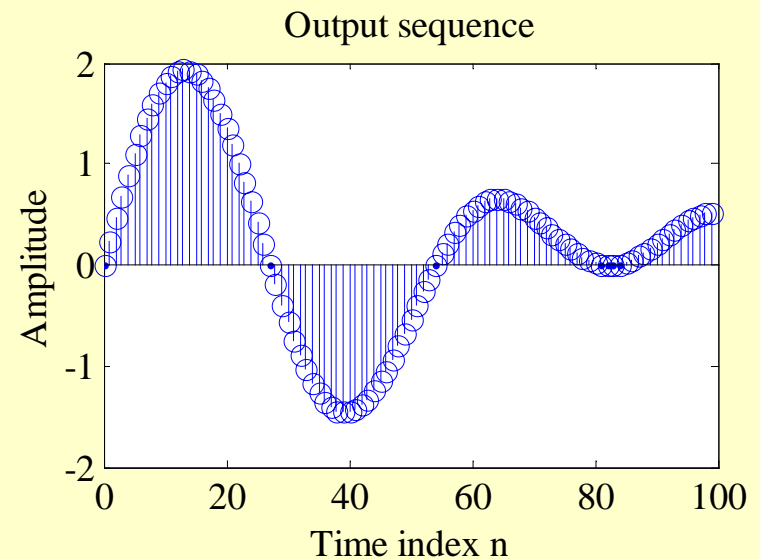
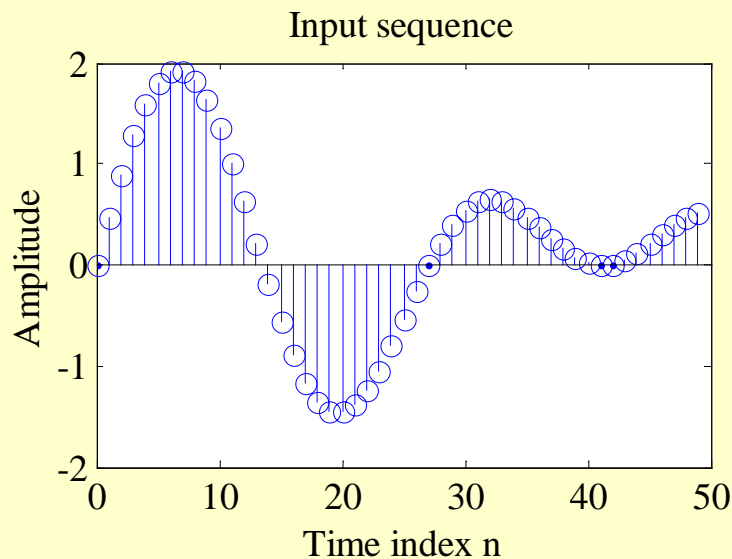
- The function `interp` can be employed to increase the sampling rate of an input signal x by an integer factor L generating the output vector y
- The lowpass filter designed by the M-file is a symmetric FIR filter

Sampling Rate Alteration Using MATLAB

- The filter allows the original input samples to appear as is in the output and finds the missing samples by minimizing the mean-square errors between these samples and their ideal values
- The interpolation of a sequence x by a factor of L can be obtained using the Program 10_6 which employs the function `interp`

Sampling Rate Alteration Using MATLAB

- Example - The input and output plots of a factor-of-2 interpolator designed using Program 10_6 are shown below

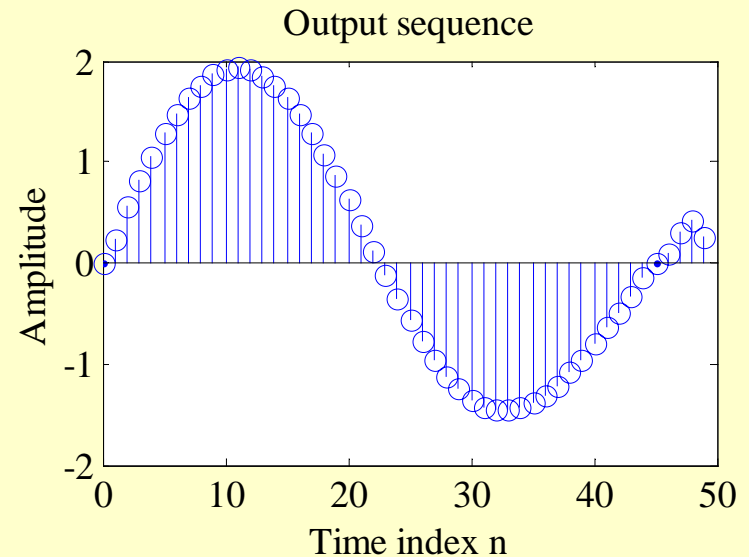
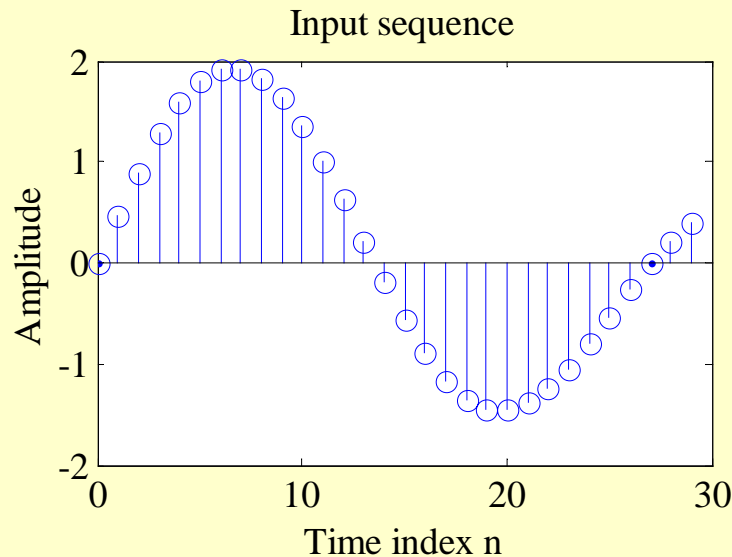


Sampling Rate Alteration Using MATLAB

- The function `resample` can be employed to increase the sampling rate of an input vector \mathbf{x} by a ratio of two positive integers, L/M , generating an output vector \mathbf{y}
- The M-file employs a lowpass FIR filter designed using `fir1` with a Kaiser window
- The fractional interpolation of a sequence can be obtained using Program 10_7 which employs the function `resample`

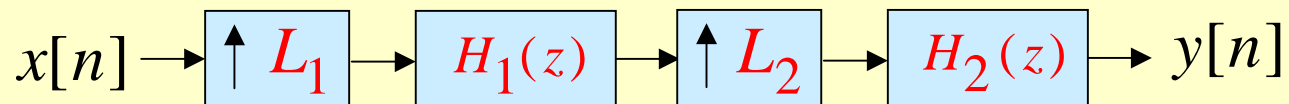
Sampling Rate Alteration Using MATLAB

- Example - The input and output plots of a factor-of-5/3 interpolator designed using Program 10_7 are given below



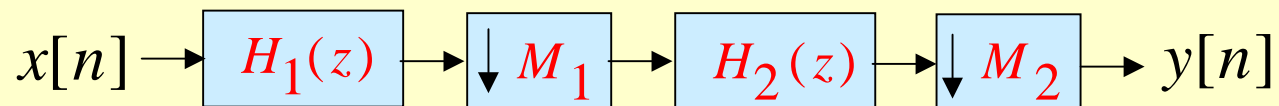
Multistage Design of Decimator and Interpolator

- The interpolator and the decimator can also be designed in more than one stages
- For example if the interpolation factor L can be expressed as a product of two integers, L_1 and L_2 , then the factor-of- L interpolator can be realized in two stages as shown below



Multistage Design of Decimator and Interpolator

- Likewise if the decimator factor M can be expressed as a product of two integers, M_1 and M_2 , then the factor-of- M interpolator can be realized in two stages as shown below

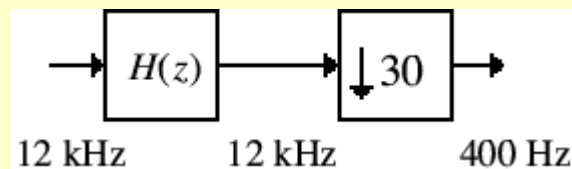


Multistage Design of Decimator and Interpolator

- Of course, the design can involve more than two stages, depending on the number of factors used to express L and M , respectively
- In general, the computational efficiency is improved significantly by designing the sampling rate alteration system as a cascade of several stages
- We consider the use of FIR filters here

Multistage Design of Decimator and Interpolator

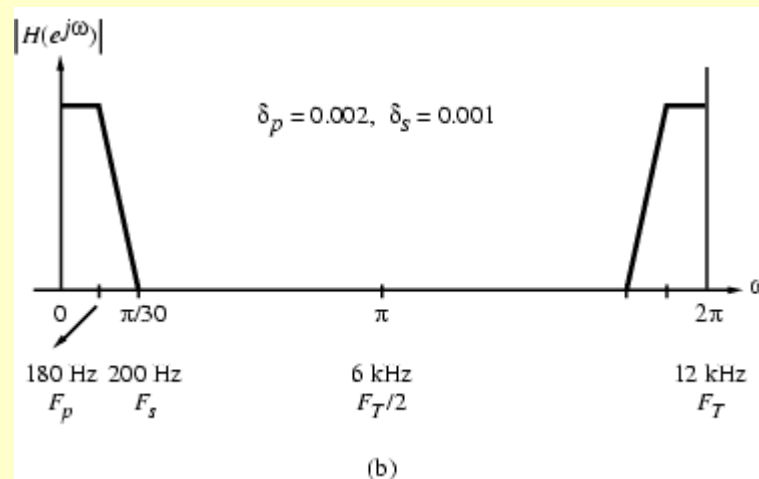
- Example - Consider the design of a decimator for reducing the sampling rate of a signal from 12 kHz to 400 Hz
- The desired down-sampling factor is therefore $M = 30$ as shown below



Multistage Design of Decimator and Interpolator

- Specifications for the decimation filter $H(z)$ are assumed to be as follows:

$$F_p = 180 \text{ Hz}, \quad F_s = 200 \text{ Hz},$$
$$\delta_p = 0.002, \quad \delta_s = 0.001$$



Multistage Design of Decimator and Interpolator

- Assume $H(z)$ to be designed as an equiripple linear-phase FIR filter
- Now Kaiser's formula for estimating the order of $H(z)$ to meet the specifications is given by

$$N = \frac{-20 \log_{10} \sqrt{\delta_p \delta_s} - 13}{14.6 \Delta f}$$

where $\Delta f = (F_s - F_p) / F_T$ is the normalized transition bandwidth

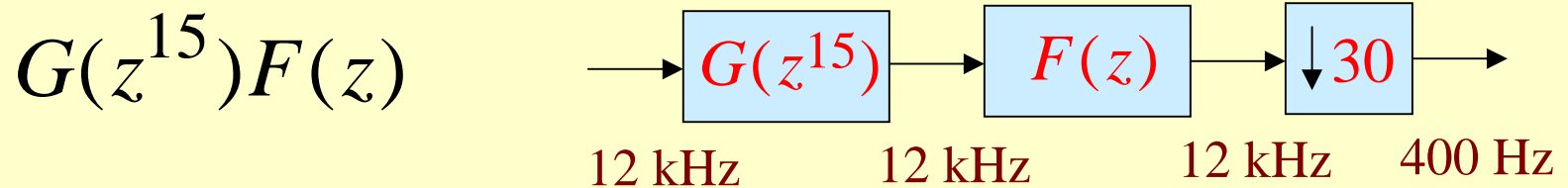
Multistage Design of Decimator and Interpolator

- Program 7_4 determines the filter order using Kaiser's formula
- Using Program 7_4 we obtain $N = 1808$
- Therefore, the number of multiplications per second in the single-stage implementation of the factor-of-30 decimator is

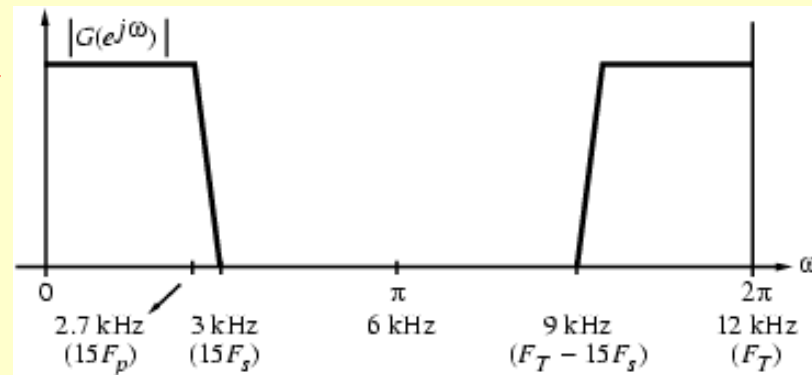
$$\mathcal{R}_{M,H} = 1809 \times \frac{12,000}{30} = 723,600$$

Multistage Design of Decimator and Interpolator

- We next implement $H(z)$ using the IFIR approach as a cascade in the form of

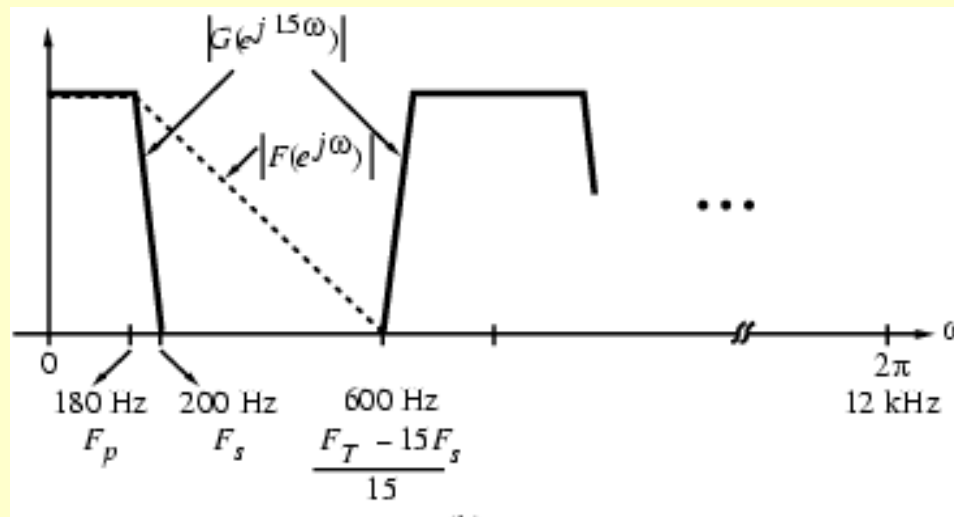


- The specifications of the parent filter $G(z)$ should thus be as shown on the right



Multistage Design of Decimator and Interpolator

- This corresponds to stretching the specifications of $H(z)$ by 15
- Figure below shows the magnitude response of $G(z^{15})$ and the desired response of $F(z)$



Multistage Design of Decimator and Interpolator

- Note: The desired response of $F(z)$ has a wider transition band as it takes into account the spectral gaps between the passbands of $G(z^{15})$
- Because of the cascade connection, the overall ripple of the cascade in dB is given by the sum of the passband ripples of $F(z)$ and $G(z^{15})$ in dB

Multistage Design of Decimator and Interpolator

- This can be compensated for by designing $F(z)$ and $G(z)$ to have a passband ripple of $\delta_p = 0.001$ each
- On the other hand, the cascade of $F(z)$ and $G(z^{15})$ has a stopband at least as good as $F(z)$ or $G(z^{15})$, individually
- So we can choose $\delta_s = 0.001$ for both filters

Multistage Design of Decimator and Interpolator

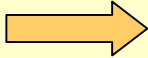
- Thus, specifications for the two filters $G(z)$ and $F(z)$ are as follows:

$$G(z): \delta_p = 0.001, \quad \delta_s = 0.001, \quad \Delta f = \frac{300}{12,000}$$

$$F(z): \delta_p = 0.001, \quad \delta_s = 0.001, \quad \Delta f = \frac{420}{12,000}$$

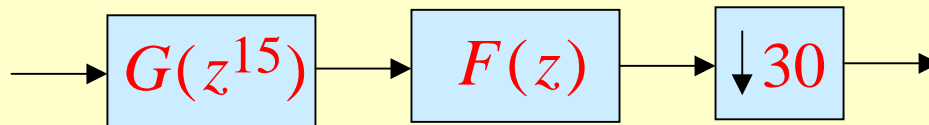
- The filter orders obtained using Program 7_4 are: Order of $G(z) = 129$
Order of $F(z) = 92$

Multistage Design of Decimator and Interpolator

- The length of $H(z)$ for a direct implementation is 1809
- The length of cascade implementation $G(z^{15})F(z)$ is $92 + 15 \times 129 + 1 = 2028$
-  The length of the cascade structure is higher

Multistage Design of Decimator and Interpolator

- The computational complexity of the decimator implemented using the cascade structure can be dramatically reduced by making use of the **cascade equivalence #1**
- To this end, we first redraw the structure

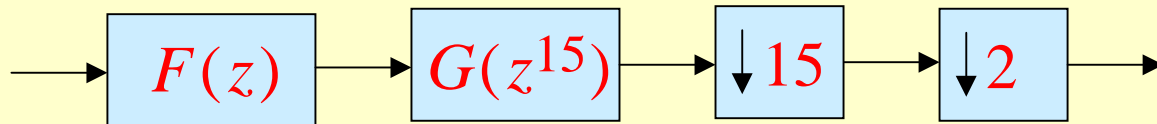


in the form shown below

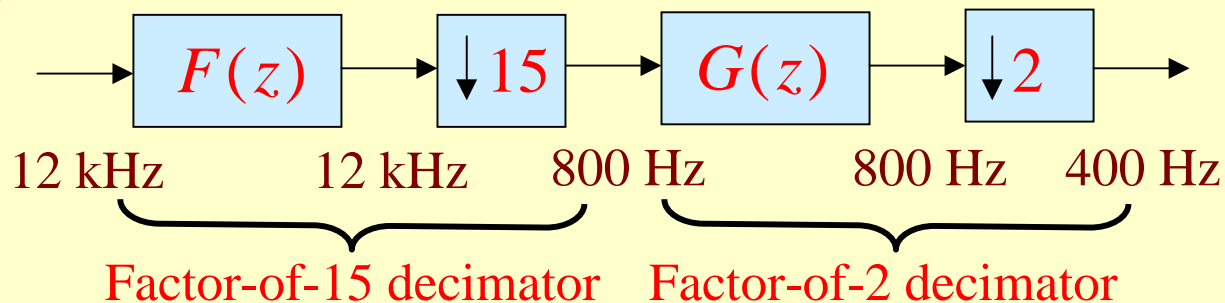


Multistage Design of Decimator and Interpolator

- The last structure is equivalent to the one shown below



- The above can be redrawn as indicated below by making use of the cascade equivalence #1



Multistage Design of Decimator and Interpolator

- From the last realization we observe that the implementation of $G(z)$ followed by a factor-of-2 down-sampler requires

$$\mathcal{R}_{M,G} = 130 \times \frac{800}{2} = 52,000 \text{ mult/sec}$$

- Likewise, the implementation of $F(z)$ followed by a factor-of-15 down-sampler requires

$$\mathcal{R}_{M,F} = 93 \times \frac{12,000}{15} = 74,400 \text{ mult/sec}$$

Multistage Design of Decimator and Interpolator

- The total complexity of the IFIR-based implementation of the factor-of-30 decimator is therefore
$$52,000 + 74,400 = 126,400 \text{ mult/sec}$$
which is about 5.72 times smaller than that of a direct implementation of the decimation filter $H(z)$