Chapter 2

2.1 **(a)** $\|x_1\|_1 = 22.85$, $\|x_1\|_2 = 9.1396$, $\|x_1\|_\infty = 4.81$,

(b) $\|x_2\|_1 = 18.68$, $\|x_2\|_2 = 7.1944$, $\|x_2\|_\infty = 3.48$.

2.2 $\mu[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases}$ Hence, $\mu[-n-1] = \begin{cases} 1, & n < 0, \\ 0, & n \geq 0. \end{cases}$ Thus, $x[n] = \mu[n] + \mu[-n-1]$.

2.3 **(a)** Consider the sequence defined by $x[n] = \sum_{k=-\infty}^{n} \delta[k]$. If $n < 0$, then $k = 0$ is not included in the sum and hence, $x[n] = 0$ for $n < 0$. On the other hand, for $n \geq 0$, $k = 0$ is included in the sum, and as a result, $x[n] = 1$ for $n \geq 0$. Therefore,

$$x[n] = \sum_{k=-\infty}^{n} \delta[k] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0, \end{cases} = \mu[n].$$

(b) Since $\mu[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0, \end{cases}$ it follows that $\mu[n-1] = \begin{cases} 1, & n \geq 1, \\ 0, & n < 1. \end{cases}$ Hence,

$$\mu[n] - \mu[n-1] = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0, \end{cases} = \delta[n].$$

2.4 Recall $\mu[n] - \mu[n-1] = \delta[n]$. Hence,

$$x[n] = \delta[n] + 3\delta[n-1] - 2\delta[n-2] + 4\delta[n-3]$$
$$= (\mu[n] - \mu[n-1]) + 3(\mu[n-1] - \mu[n-2]) - 2(\mu[n-2] - \mu[n-3]) + 4(\mu[n-3] - \mu[n-4])$$
$$= \mu[n] + 2\mu[n-1] - 5\mu[n-2] + 6\mu[n-3] - 4\mu[n-4].$$

2.5 **(a)** $c[n] = x[-n+2] = \{2 \quad \underset{\uparrow}{0} \quad -3 \quad -2 \quad 1 \quad 5 \quad -4\}$,

(b) $d[n] = y[-n-3] = \{-2 \quad 7 \quad 8 \quad 0 \quad -1 \quad -3 \quad 6 \quad 0 \quad \underset{\uparrow}{0}\}$,

(c) $e[n] = w[-n] = \{5 \quad -2 \quad 0 \quad -1 \quad 2 \quad 2 \quad 3 \quad 0 \quad \underset{\uparrow}{0}\}$,

(d) $u[n] = x[n] + y[n-2] = \{-4 \quad 5 \quad 1 \quad \underset{\uparrow}{-2} \quad 3 \quad -3 \quad 1 \quad 0 \quad 8 \quad 7 \quad -2\}$,

(e) $v[n] = x[n] \cdot w[n+4] = \{0 \quad 15 \quad 2 \quad \underset{\uparrow}{-4} \quad 3 \quad 0 \quad -4 \quad 0\}$,

(f) $s[n] = y[n] - w[n+4] = \{-3 \quad 4 \quad \underset{\uparrow}{-5} \quad 0 \quad 0 \quad 10 \quad 2 \quad -2\}$,

(g) $r[n] = 3.5\, y[n] = \{21 \quad \underset{\uparrow}{-10.5} \quad -3.5 \quad 0 \quad 2.8 \quad 24.5 \quad -7\}$.

2.6 **(a)** $x[n] = -4\delta[n+3] + 5\delta[n+2] + \delta[n+1] - 2\delta[n] - 3\delta[n-1] + 2\delta[n-3]$,

$y[n] = 6\delta[n+1] - 3\delta[n] - \delta[n-1] + 8\delta[n-3] + 7\delta[n-4] - 2\delta[n-5]$,

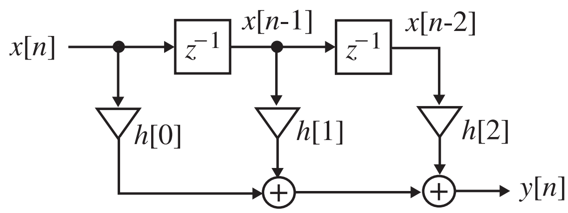
$w[n] = 3\delta[n-2] + 2\delta[n-3] + 2\delta[n-4] - \delta[n-5] - 2\delta[n-7] + 5\delta[n-8]$,

(b) Recall $\delta[n] = \mu[n] - \mu[n-1]$. Hence,

$$x[n] = -4(\mu[n+3] - \mu[n+2]) + 5(\mu[n+2] - \mu[n+1]) + (\mu[n+1] - \mu[n])$$
$$-2(\mu[n] - \mu[n-1]) - 3(\mu[n-1] - \mu[n-2]) + 2(\mu[n-3] - \mu[n-4])$$

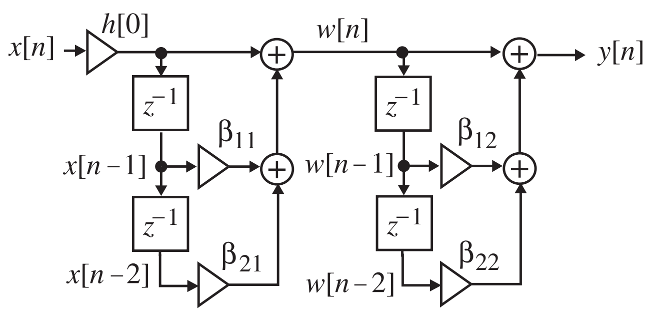
$= -4\mu[n+3] + 9\mu[n+2] - 4\mu[n+1] - 3\mu[n] - \mu[n-1] + 3\mu[n-2] + 2\mu[n-3] - 2\mu[n-4],$

2.7 **(a)**



From the above figure it follows that $y[n] = h[0]x[n] + h[1]x[n-1] + h[2]x[n-2]$.

(b)

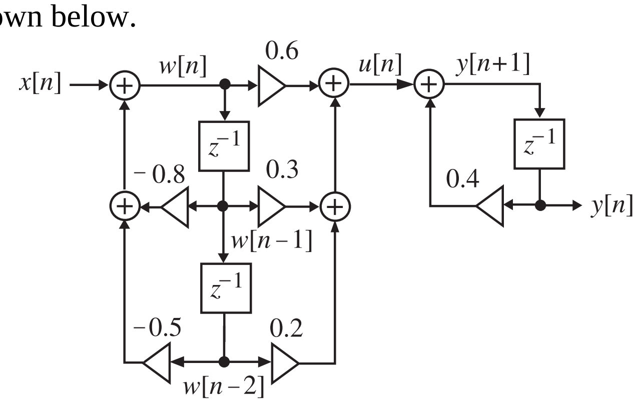


From the above figure we get $w[n] = h[0](x[n] + \beta_{11}x[n-1] + \beta_{21}x[n-2])$ and $y[n] = w[n] + \beta_{12}w[n-1] + \beta_{22}w[n-2]$. Making use of the first equation in the second we arrive at

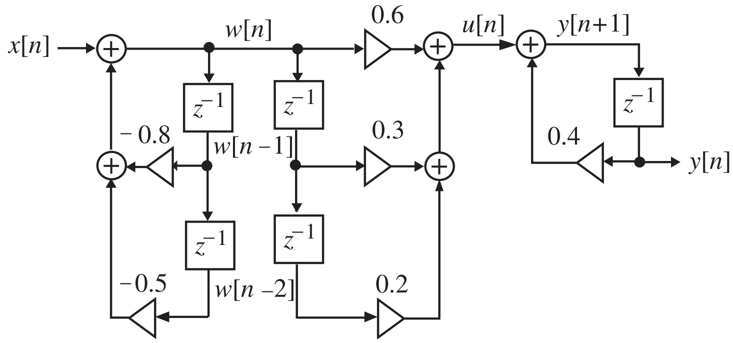
$$y[n] = h[0](x[n] + \beta_{11}x[n-1] + \beta_{21}x[n-2]) \\ + \beta_{12}h[0](x[n-1] + \beta_{11}x[n-2] + \beta_{21}x[n-3]) \\ + \beta_{22}h[0](x[n-2] + \beta_{11}x[n-3] + \beta_{21}x[n-4])$$

$$= h[0]\big(x[n] + (\beta_{11} + \beta_{12})x[n-1] + (\beta_{21} + \beta_{12}\beta_{11} + \beta_{22})x[n-2] \\ + (\beta_{12}\beta_{21} + \beta_{22}\beta_{11})x[n-3] + \beta_{22}\beta_{21}x[n-4]\big).$$

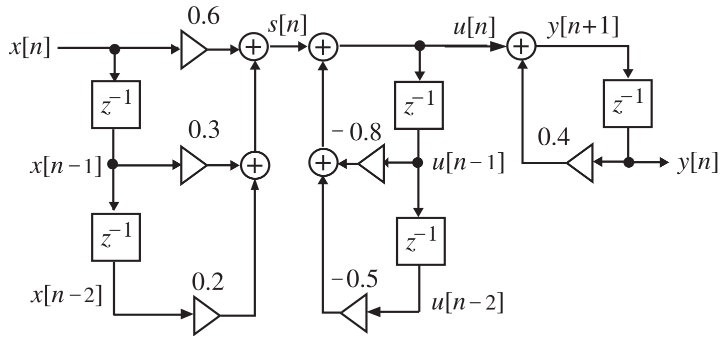
(c) Figure P2.1(c) is a cascade of a first-order section and a second-order section. The input-output relation remains unchanged if the ordering of the two sections is interchanged as shown below.



The second-order section can be redrawn as shown below without changing its input-output relation.



The second-order section can be seen to be cascade of two sections. Interchanging their ordering we finally arrive at the structure shown below:



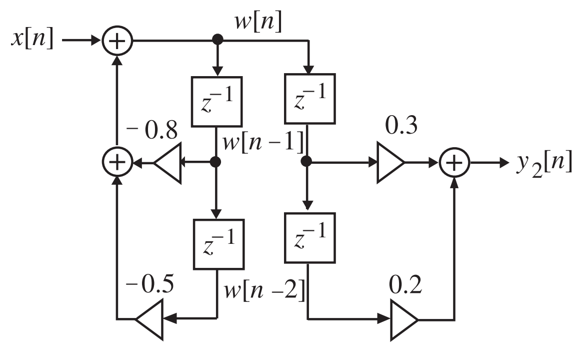
Analyzing the above structure we arrive at

$$s[n] = 0.6x[n] + 0.3x[n-1] + 0.2x[n-2],$$
$$u[n] = s[n] - 0.8u[n-1] - 0.5u[n-2],$$
$$y[n+1] = u[n] + 0.4y[n].$$

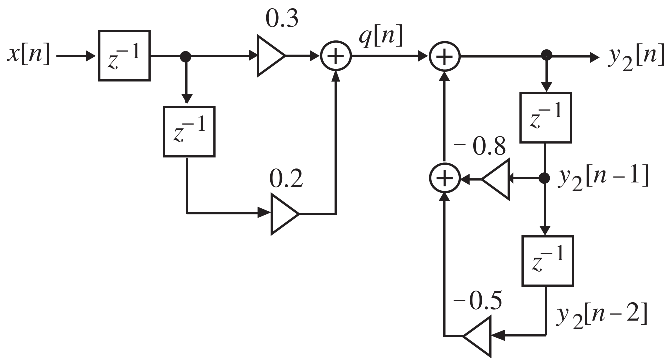
From $u[n] = y[n+1] - 0.4y[n]$. Substituting this in the second equation we get after some algebra $y[n+1] = s[n] - 0.4y[n] - 0.18y[n-1] + 0.8y[n-2]$. Making use of the first equation in this equation we finally arrive at the desired input-output relation

$$y[n] + 0.4y[n-1] + 0.18y[n-2] - 0.2y[n-3] = 0.6x[n-1] + 0.3x[n-2] + 0.2x[n-3].$$

(d) Figure P2.19(d) is a parallel connection of a first-order section and a second-order section. The second-order section can be redrawn as a cascade of two sections as indicated below:



Interchanging the order of the two sections we arrive at an equivalent structure shown below:



Analyzing the above structure we get

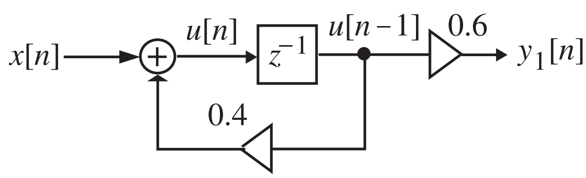
$$q[n] = 0.3x[n-1] + 0.2x[n-2],$$

$$y_2[n] = q[n] - 0.8y_2[n-1] - 0.5y_2[n-2].$$

Substituting the first equation in the second we have

$$y_2[n] + 0.8y_2[n-1] + 0.5y_2[n-2] = 0.3x[n-1] + 0.2x[n-2]. \qquad (2\text{-}1)$$

Analyzing the first-order section of Figure P2.1(d) given below



we get

$$u[n] = x[n] + 0.4u[n-1],$$

$$y_1[n] = 0.6u[n-1].$$

Solving the above two equations we have

$$y_1[n] - 0.4y_1[n-1] = 0.6x[n-1]. \qquad (2\text{-}2)$$

The output $y[n]$ of the structure of Figure P2.19(d) is given by

$$y[n] = y_1[n] + y_2[n]. \qquad (2\text{-}3)$$

From Eq. (2-2) we get $0.8y_1[n-1] - 0.32y_1[n-2] = 0.48\,x[n-2]$ and $0.5y_1[n-2] - 0.2y_1[n-3] = 0.3\,x[n-3]$. Adding the last two equations to Eq. (2-2) we arrive at $y_1[n] + 0.4y_1[n-1] + 0.18y_1[n-2] - 0.2\,y_1[n-3]$

$$= 0.6\,x[n-1] + 0.48\,x[n-2] + 0.3x[n-3]. \qquad (2\text{-}4)$$

Similarly, from Eq. (2-1) we get $-0.4y_2[n-1] - 0.32y_2[n-2] - 0.2y_2[n-3] = -0.12x[n-2] - 0.08x[n-3]$. Adding this equation to Eq. (2-1) we arrive at

$$y_2[n] + 0.4y_2[n-1] + 0.18y_2[n-2] - 0.2\,y_2[n-3]$$

$$= 0.3x[n-1] + 0.08x[n-2] - 0.08\,x[n-3]. \qquad (2\text{-}5)$$

Adding Eqs. (2-4) and (2-5), and making use of Eq. (2-3) we finally arrive at the input-output relation of Figure P2.1(d) as:

$$y[n] + 0.4y[n-1] + 0.18y[n-2] - 0.2\,y[n-3] = 0.9\,x[n-1] + 0.56\,x[n-2] + 0.22\,x[n-3].$$

2.8 **(a)** $x_1^*[n] = \{1-j4 \quad -2-j5 \quad \underset{\uparrow}{3+j2} \quad -7-j3 \quad -1-j\},$

$x_1^*[-n] = \{-1-j \quad -7-j3 \quad \underset{\uparrow}{3+j2} \quad -2-j5 \quad 1-j4\}.$ Therefore

$x_{1,cs}[n] = \frac{1}{2}\left(x_1^*[n] + x_1^*[-n]\right) = \{j1.5 \quad -4.5+j \quad \underset{\uparrow}{3} \quad -4.5-j \quad -j1.5\},$

$x_{1,ca}[n] = \frac{1}{2}\left(x_1^*[n] - x_1^*[-n]\right) = \{1+j2.5 \quad 2.5+j4 \quad \underset{\uparrow}{-j2} \quad -2.5+j4 \quad -1+j2.5\}.$

(b) $x_2[n] = e^{j\pi n/3}$. Hence, $x_2^*[n] = e^{-j\pi n/3}$ and thus, $x_2^*[-n] = e^{j\pi n/3} = x_2[n]$. Therefore, $x_{2,cs}[n] = \frac{1}{2}\left(x_2^*[n] + x_2^*[-n]\right) = e^{j2\pi n/3} = x_2[n]$, and $x_{2,ca}[n] = \frac{1}{2}\left(x_2^*[n] - x_2^*[-n]\right) = 0.$

(c) $x_3[n] = j\,e^{-j\pi n/5}$. Hence, $x_3^*[n] = -j\,e^{j\pi n/5}$ and thus, $x_3^*[-n] = -j\,e^{-j\pi n/5} = -x_3[n]$. Therefore, $x_{3,cs}[n] = \frac{1}{2}\left(x_3^*[n] + x_3^*[-n]\right) = 0$, and $x_{3,ca}[n] = \frac{1}{2}\left(x_3^*[n] - x_3^*[-n]\right) = x_3[n] = j\,e^{-j\pi n/5}$.

2.9 **(a)** $x[n] = \{-4 \quad 5 \quad 1 \quad \underset{\uparrow}{-2} \quad -3 \quad 0 \quad 2\}$. Hence, $x[-n] = \{2 \quad 0 \quad -3 \quad \underset{\uparrow}{-2} \quad 1 \quad 5 \quad -4\}$.

Therefore, $x_{ev}[n] = \frac{1}{2}(x[n] + x[-n]) = \frac{1}{2}\{-2 \quad 5 \quad -2 \quad \underset{\uparrow}{-4} \quad -2 \quad 5 \quad -2\}$

$= \{-1 \quad 2.5 \quad -1 \quad \underset{\uparrow}{-2} \quad -1 \quad 2.5 \quad -1\}$

and $x_{od}[n] = \frac{1}{2}(x[n] - x[-n]) = \frac{1}{2}\{-6 \quad 5 \quad 4 \quad \underset{\uparrow}{0} \quad -4 \quad -5 \quad 6\}$

$= \{-3 \quad 2.5 \quad 2 \quad \underset{\uparrow}{0} \quad -2 \quad -2.5 \quad 3\}.$

(b) $y[n] = \{0 \quad 0 \quad 0 \quad 0 \quad 6 \quad \underset{\uparrow}{-3} \quad -1 \quad 0 \quad 8 \quad 7 \quad -2\}$. Hence,

$y[-n] = \{-2 \quad 7 \quad 8 \quad 0 \quad -1 \quad \underset{\uparrow}{-3} \quad 6 \quad 0 \quad 0 \quad 0 \quad 0\}.$

Therefore, $y_{ev}[n] = \frac{1}{2}(y[n] + y[-n]) = \{-1 \quad 3.5 \quad 4 \quad 0 \quad 2.5 \quad \underset{\uparrow}{-3} \quad 2.5 \quad 0 \quad 4 \quad 3.5 \quad -1\}$

and $y_{od}[n] = \frac{1}{2}(y[n] - y[-n]) = \{1 \quad -3.5 \quad -4 \quad 0 \quad 3.5 \quad \underset{\uparrow}{0} \quad -3.5 \quad 0 \quad 4 \quad 3.5 \quad -1\}.$

(c) $w[n] = \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \underset{\uparrow}{0} \quad 0 \quad 3 \quad 2 \quad 2 \quad -1 \quad 0 \quad -2 \quad 5\}$. Hence,

$w[-n] = \{5 \quad -2 \quad 0 \quad -1 \quad 2 \quad 2 \quad 3 \quad 0 \quad \underset{\uparrow}{0} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\}$. Therefore

$w_{ev}[n] = \frac{1}{2}(w[n] + w[-n])$

$$= \{2.5 \quad -1 \quad 0 \quad -0.5 \quad 1 \quad 1 \quad 1.5 \quad 0 \quad \underset{\uparrow}{0} \quad 0 \quad 1.5 \quad 1 \quad 1 \quad -0.5 \quad 0 \quad -1 \quad 2.5\}$$ and

$$w_{od}[n] = \frac{1}{2}(w[n] - w[-n])$$
$$= \{-2.5 \quad 1 \quad 0 \quad 0.5 \quad -1 \quad -1 \quad -1.5 \quad 0 \quad \underset{\uparrow}{0} \quad 0 \quad 1.5 \quad 1 \quad 1 \quad -0.5 \quad 0 \quad -1 \quad 2.5\}.$$

2.10 **(a)** $x_1[n] = \mu[n+2]$. Hence, $x_1[-n] = \mu[-n+2]$. Therefore,

$$x_{1,ev}[n] = \frac{1}{2}(\mu[n+2] + \mu[-n+2]) = \begin{cases} 1/2, & n \geq 3, \\ 1, & -2 \leq n \leq 2, \\ 1/2, & -3 \leq n, \end{cases}$$ and

$$x_{1,od}[n] = \frac{1}{2}(\mu[n+2] - \mu[-n+2]) = \begin{cases} 1/2, & n \geq 3, \\ 0, & -2 \leq n \leq 2, \\ -1/2, & -3 \leq n. \end{cases}$$

(b) $x_2[n] = \alpha^n \mu[n-3]$. Hence, $x_2[-n] = \alpha^{-n}\mu[-n-3]$. Therefore,

$$x_{2,ev}[n] = \frac{1}{2}\left(\alpha^n \mu[n-3] + \alpha^{-n}\mu[-n-3]\right) = \begin{cases} \frac{1}{2}\alpha^n, & n \geq 3, \\ 0, & -2 \leq n \leq 2, \\ \frac{1}{2}\alpha^{-n}, & -3 \leq n, \end{cases}$$ and

$$x_{2,od}[n] = \frac{1}{2}\left(\alpha^n \mu[n-3] - \alpha^{-n}\mu[-n-3]\right) = \begin{cases} \frac{1}{2}\alpha^n, & n \geq 3, \\ 0, & -2 \leq n \leq 2, \\ -\frac{1}{2}\alpha^{-n}, & -3 \leq n. \end{cases}$$

(c) $x_3[n] = n\alpha^n \mu[n]$. Hence, $x_3[-n] = -n\alpha^{-n}\mu[-n]$. Therefore,

$$x_{3,ev}[n] = \frac{1}{2}\left(n\alpha^n \mu[n] + (-n)\alpha^{-n}\mu[-n]\right) = \frac{1}{2}|n|\alpha^{|n|}$$ and

$$x_{3,od}[n] = \frac{1}{2}\left(n\alpha^n \mu[n] - (-n)\alpha^{-n}\mu[-n]\right) = \frac{1}{2}n\alpha^{|n|}.$$

(d) $x_4[n] = \alpha^{|n|}$. Hence, $x_4[-n] = \alpha^{|-n|} = \alpha^{|n|} = x_4[n]$. Therefore,

$$x_{4,ev}[n] = \frac{1}{2}(x_4[n] + x_4[-n]) = \frac{1}{2}(x_4[n] + x_4[n]) = x_4[n] = \alpha^{|n|}$$ and

$$x_{4,od}[n] = \frac{1}{2}(x_4[n] - x_4[-n]) = \frac{1}{2}(x_4[n] - x_4[n]) = 0.$$

2.11 $x_{ev}[n] = \frac{1}{2}\left(x[n] + x[-n]\right)$. Thus, $x_{ev}[-n] = \frac{1}{2}\left(x[-n] + x[n]\right) = x_{ev}[n]$. Hence, $x_{ev}[n]$ is an even sequence. Likewise, $x_{od}[n] = \frac{1}{2}\left(x[n] - x[-n]\right)$. Thus, $x_{od}[-n] = \frac{1}{2}\left(x[-n] - x[n]\right) = -x_{od}[n]$. Hence, $x_{od}[n]$ is an odd sequence.

2.12 **(a)** $g[n] = x_{ev}[n]x_{ev}[n]$. Thus, $g[-n] = x_{ev}[-n]x_{ev}[-n] = x_{ev}[n]x_{ev}[n] = g[n]$. Hence, $g[n]$ is an even sequence.

(b) $u[n] = x_{ev}[n]x_{od}[n]$. Thus, $u[-n] = x_{ev}[-n]x_{od}[-n] = x_{ev}[n](-x_{od}[n]) = -u[n]$. Hence, $u[n]$ is an odd sequence.

(c) $v[n] = x_{od}[n]x_{od}[n]$. Thus, $v[-n] = x_{od}[-n]x_{od}[-n] = (-x_{od}[n])(-x_{od}[n])$ $= x_{od}[n]x_{od}[n] = v[n]$. Hence, $v[n]$ is an even sequence.

2.13 **(a)** Since $x[n]$ is causal, $x[n] = 0,\ n < 0$. Also, $x[-n] = 0,\ n > 0$. Now, $x_{ev}[n] = \frac{1}{2}(x[n] + x[-n])$. Hence, $x_{ev}[0] = \frac{1}{2}(x[0] + x[0]) = x[0]$ and $x_{ev}[n] = \frac{1}{2}x[n], n > 0$. Combining the two equations we get $x[n] = \begin{cases} 2x_{ev}[n], & n > 0, \\ x_{ev}[n], & n = 0, \\ 0, & n < 0. \end{cases}$

Likewise, $x_{od}[n] = \frac{1}{2}(x[n] - x[-n])$. Hence, $x_{od}[0] = \frac{1}{2}(x[0] - x[0]) = 0$ and $x_{od}[n] = \frac{1}{2}x[n], n > 0$. Combining the two equations we get $x[n] = \begin{cases} 2x_{ev}[n], & n > 0, \\ 0, & n \le 0. \end{cases}$

(b) Since $y[n]$ is causal, $y[n] = 0,\ n < 0$. Also, $y[-n] = 0,\ n > 0$. Let $y[n] = y_{re}[n] + jy_{im}[n]$, where $y_{re}[n]$ and $y_{im}[n]$ are real causal sequences.

Now, $y_{ca}[n] = \frac{1}{2}\left(y[n] - y^*[-n]\right)$. Hence, $y_{ca}[0] = \frac{1}{2}\left(y[0] - y^*[0]\right) = jy_{im}[0]$ and $y_{ca}[n] = \frac{1}{2}y[n], n > 0$. Since $y_{re}[0]$ is not known, $y[n]$ cannot be fully recovered from $y_{ca}[n]$.

Likewise, $y_{cs}[n] = \frac{1}{2}\left(y[n] + y^*[-n]\right)$. Hence, $y_{cs}[0] = \frac{1}{2}\left(y[0] + y^*[0]\right) = y_{re}[0]$ and $y_{cs}[n] = \frac{1}{2}y[n], n > 0$. Since $y_{im}[0]$ is not known, $y[n]$ cannot be fully recovered from $y_{cs}[n]$.

2.14 Since $x[n]$ is causal, $x[n] = 0,\ n < 0$. From the solution of Problem 2.13 we have

$$x[n] = \begin{cases} 2x_{ev}[n], & n > 0, \\ x_{ev}[n], & n = 0, \\ 0, & n < 0, \end{cases} = \begin{cases} 2\cos(\omega_o n), & n > 0, \\ 1 & n = 0, \\ 0, & n < 0, \end{cases} = 2\cos(\omega_o n)\mu[n] - \delta[n].$$

2.15 **(a)** $\{x[n]\} = \{A\alpha^n\}$ where A and α are complex numbers with $|\alpha| < 1$. Since for $n < 0$, $|\alpha|^n$ can become arbitrarily large, $\{x[n]\}$ is not a bounded sequence.

(b) $y[n] = A\alpha^n \mu[n] = \begin{cases} A\alpha^n, & n \geq 0, \\ 0, & n < 0, \end{cases}$ where A and α are complex numbers with $|\alpha| < 1$. Here, $|\alpha|^n \leq 1, n \geq 0$. Hence $|y[n]| \leq |A|$ for all values of n. Hence, $\{y[n]\}$ is a bounded sequence.

(c) $\{h[n]\} = C\beta^n \mu[n]$ where C and β are complex numbers with $|\beta| > 1$. Since for $n > 0$, $|\beta|^n$ can become arbitrarily large, $\{h[n]\}$ is not a bounded sequence.

(d) $\{g[n]\} = 4\cos(\omega_o n)$. Since $|g[n]| \leq 4$ for all values of $n, \{g[n]\}$ is a bounded sequence.

(e) $v[n] = \begin{cases} \left(1 - \frac{1}{n^2}\right), & n \geq 1, \\ 0, & n \leq 0. \end{cases}$ Since $\frac{1}{n^2} < 1$ for $n > 1$ and $\frac{1}{n^2} = 1$ for $n = 1$, $|v[n]| < 1$ for all values of n. Thus $\{v[n]\}$ is a bounded sequence.

2.16 $x[n] = \frac{(-1)^{n+1}}{n} \mu[n-1]$. Now $\sum_{n=-\infty}^{\infty} |x[n]| = \sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$. Hence $\{x[n]\}$ is not absolutely summable.

2.17 **(a)** $x_1[n] = \alpha^n \mu[n-1]$. Now $\sum_{n=-\infty}^{\infty} |x_2[n]| = \sum_{n=1}^{\infty} |\alpha^n| = \sum_{n=1}^{\infty} |\alpha|^n = \frac{|\alpha|}{1-|\alpha|} < \infty$, since $|\alpha| < 1$. Hence, $\{x_1[n]\}$ is absolutely summable.

(b) $x_2[n] = \alpha^n \mu[n-1]$. Now $\sum_{n=-\infty}^{\infty} |x_2[n]| = \sum_{n=1}^{\infty} |n\alpha^n| = \sum_{n=1}^{\infty} n|\alpha|^n = \frac{|\alpha|}{(1-|\alpha|)^2} < \infty$, since $|\alpha|^2 < 1$. Hence, $\{x_2[n]\}$ is absolutely summable.

(c) $x_3[n] = n^2 \alpha^n \mu[n-1]$. Now $\sum_{n=-\infty}^{\infty} |x_3[n]| = \sum_{n=1}^{\infty} |n^2 \alpha^n| = \sum_{n=1}^{\infty} n^2 |\alpha|^n$

$$= |\alpha| + 2^2|\alpha|^2 + 3^2|\alpha|^3 + 4^2|\alpha|^4 + \ldots$$

$$= (|\alpha| + |\alpha|^2 + |\alpha|^3 + |\alpha|^4 + \ldots) + 3(|\alpha|^2 + |\alpha|^3 + |\alpha|^4 + \ldots) + 5(|\alpha|^3 + |\alpha|^4 + |\alpha|^5 + \ldots)$$

$$+7(|\alpha|^4+|\alpha|^5+|\alpha|^6+\ldots) = \frac{|\alpha|}{1-|\alpha|}+\frac{3|\alpha|^2}{1-|\alpha|}+\frac{5|\alpha|^3}{1-|\alpha|}+\frac{7|\alpha|^4}{1-|\alpha|}+\ldots$$

$$=\frac{1}{1-|\alpha|}\left(\sum_{n=1}^{\infty}(2n-1)|\alpha|^n\right)=\frac{1}{1-|\alpha|}\left(2\sum_{n=1}^{\infty}n|\alpha|^n-\sum_{n=1}^{\infty}|\alpha|^n\right)=\frac{1}{1-|\alpha|}\left(\frac{2|\alpha|}{(1-|\alpha|)^2}-\frac{|\alpha|}{1-|\alpha|}\right)$$

$=\dfrac{|\alpha|(1+|\alpha|)}{(1-|\alpha|)^3}<\infty.$ Hence, $\{x_3[n]\}$ is absolutely summable.

2.18 **(a)** $x_a[n]=\dfrac{1}{2^n}\mu[n]$. Now $\displaystyle\sum_{n=-\infty}^{\infty}|x_a[n]|=\sum_{n=0}^{\infty}\left|\frac{1}{2^n}\right|=\sum_{n=0}^{\infty}\frac{1}{2^n}=\frac{1}{1-\frac{1}{2}}=2<\infty$. Hence, $\{x_a[n]\}$ is absolutely summable.

(b) $x_b[n]=\dfrac{1}{(n+1)(n+2)}\mu[n]$. Now $\displaystyle\sum_{n=-\infty}^{\infty}|x_b[n]|=\sum_{n=0}^{\infty}\left|\frac{1}{(n+1)(n+2)}\right|$

$\displaystyle=\sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\ldots=1<\infty.$ Hence, $\{x_b[n]\}$ is absolutely summable.

2.19 **(a)** A sequence $x[n]$ is absolutely summable if $\displaystyle\sum_{n=-\infty}^{\infty}|x[n]|<\infty$. By Schwartz inequality we have $\displaystyle\sum_{n=-\infty}^{\infty}|x[n]|^2\le\left(\sum_{n=-\infty}^{\infty}|x[n]|\right)\left(\sum_{n=-\infty}^{\infty}|x[n]|\right)<\infty$. Hence, an absolutely summable sequence is square summable and has thus finite energy.

Now consider the sequence $x[n]=\frac{1}{n}\mu[n-1]$. The convergence of an infinite series can be shown via the integral test. Let $a_n=f(x)$, where a continuous, positive and decreasing function is for all $x\ge 1$. Then the series $\displaystyle\sum_{n=1}^{\infty}a_n$ and the integral $\displaystyle\int_1^{\infty}f(x)dx$ both converge or both diverge. For $a_n=\frac{1}{n}, f(x)=\frac{1}{x}$. But $\displaystyle\int_1^{\infty}\frac{1}{x}dx=(\ln x)\Big|_1^{\infty}=\infty-0=\infty$. Hence, $\displaystyle\sum_{n=-\infty}^{\infty}|x[n]|=\sum_{n=1}^{\infty}\frac{1}{n}$ does not converge. As a result, $x[n]=\frac{1}{n}\mu[n-1]$ is not absolutely summable.

(b) To show that $\{x[n]\}$ is square-summable, we observe that here $a_n = \frac{1}{n^2}$, and thus, $f(x) = \frac{1}{x^2}$. Now, $\int_1^{\infty} \frac{1}{x^2} dx = \left(-\frac{1}{x}\right)\Big|_1^{\infty} = -\frac{1}{\infty} + 1 = 1$. Hence, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, or in other words, $x[n] = \frac{1}{n}\mu[n-1]$ is square-summable.

2.20 See Problem 2.19, Part (a) solution.

2.21 $x_2[n] = \frac{\cos\omega_c n}{\pi n}\mu[n-1]$. Now, $\sum_{n=-\infty}^{\infty} |x_2[n]|^2 = \sum_{n=1}^{\infty}\left(\frac{\cos\omega_c n}{\pi n}\right)^2 \le \sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2}$. Since, $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$, $\sum_{n=1}^{\infty}\left(\frac{\cos\omega_c n}{\pi n}\right)^2 \le \frac{1}{6}$. Therefore $x_2[n]$ is square-summable.

Using the integral test (See Problem 2.19, Part (a) solution) we now show that $x_2[n]$ is not absolutely summable. Now, $\int_1^{\infty}\left|\frac{\cos\omega_c x}{\pi x}\right| dx = \frac{1}{\pi}\cdot\frac{\left|\frac{\cos\omega_c x}{\pi x}\right|}{\cos\omega_c x}\cdot\cos\mathrm{int}(\omega_c x)\Bigg|_1^{\infty}$ where $\cos\mathrm{int}$ is the cosine integral function. Since $\int_1^{\infty}\left|\frac{\cos\omega_c x}{\pi x}\right| dx$ diverges, $\sum_{n=1}^{\infty}\left|\frac{\cos\omega_c n}{\pi n}\right|$ also diverges. Hence, $x_2[n]$ is not absolutely summable.

2.22 $\mathcal{P}_x = \lim_{K\to\infty}\frac{1}{2K+1}\sum_{n=-K}^{K} x^2[n] = \lim_{K\to\infty}\frac{1}{2K+1}\sum_{K=-\infty}^{\infty}(x_{ev}[n] + x_{od}[n])^2$

$= \lim_{K\to\infty}\frac{1}{2K+1}\sum_{n=-K}^{K}\left(x_{ev}^2[n] + x_{od}^2[n] + 2x_{ev}[n]x_{od}[n]\right)$

$= \mathcal{P}_{x_{ev}} + \mathcal{P}_{x_{od}} + \frac{1}{2}\lim_{K\to\infty}\frac{1}{2K+1}\sum_{n=-K}^{K}(x[n] + x[-n])(x[n] - x[-n])$

$= \mathcal{P}_{x_{ev}} + \mathcal{P}_{x_{od}} + \lim_{K\to\infty}\frac{1}{2K+1}\cdot\frac{1}{2}\left(\sum_{n=-K}^{K} x^2[n] - \sum_{n=-\infty}^{\infty} x^2[-n]\right) = \mathcal{P}_{x_{ev}} + \mathcal{P}_{x_{od}}$

as $\sum_{n=-K}^{K} x^2[n] = \sum_{n=-K}^{K} x^2[-n]$. Now for the given sequence,

$\mathcal{P}_{x_{od}} = \lim_{K\to\infty}\frac{1}{2K+1}\sum_{n=-K}^{K} x_{od}^2[n] = \lim_{K\to\infty}\frac{1}{2K+1}\sum_{n=0}^{K}\left(\frac{1}{3}\right)^6 = \left(\frac{1}{3}\right)^6\lim_{K\to\infty}\frac{1}{2K+1}\sum_{n=0}^{K} 1$

$= \left(\frac{1}{3}\right)^6\lim_{K\to\infty}\frac{K+1}{2K+1} = \frac{1}{2}\left(\frac{1}{3}\right)^6$. Hence, $\mathcal{P}_{x_{ev}} = \mathcal{P}_x - \mathcal{P}_{x_{od}} = 10 - \frac{1}{2}\left(\frac{1}{3}\right)^6$.

2.23 $x[n] = \sin(2\pi kn/N),\ 0 \le n \le N-1$. Now $\mathcal{E}_x = \sum_{n=0}^{N-1} |x[n]|^2 = \sum_{n=0}^{N-1} \sin^2(2\pi kn/N)$

$= \frac{1}{2}\sum_{n=0}^{N-1}\big(1-\cos(4\pi kn/N)\big) = \frac{N}{2} - \frac{1}{2}\sum_{n=0}^{N-1}\cos(4\pi kn/N)$. Let $C = \sum_{n=0}^{N-1}\cos(4\pi kn/N)$ and

$S = \sum_{n=0}^{N-1}\sin(4\pi kn/N)$. Then $C + jS = \sum_{n=0}^{N-1} e^{-j4\pi kn/N} = \frac{1-e^{-j4\pi kn}}{1-e^{-j4\pi k/N}} = 0$. This implies

$C = 0$. Hence $\mathcal{E}_x = \frac{N}{2}$.

2.24 **(a)** $x[n] = A^{\alpha}\mu[n]$. Then $\mathcal{E}_{x_a} = \sum_{n=-\infty}^{\infty} |x_a[n]|^2 = A^2 \sum_{n=0}^{\infty}\alpha^{2n} = \frac{A^2}{1-\alpha^2}$.

(b) $x_b[n] = \frac{1}{n^2}\mu[n-1]$. Then $\mathcal{E}_{x_b} \sum_{n=-\infty}^{\infty} |x_b[n]|^2 = \sum_{n=1}^{\infty}\left|\frac{1}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$.

2.25 **(a)** $x_1[n] = (-1)^n$. Then average power

$\mathcal{P}_{x_1} = \lim_{K\to\infty}\frac{1}{2K+1}\sum_{n=-K}^{K}|x_1[n]|^2 = \lim_{K\to\infty}\frac{1}{2K+1}(2K+1) = 1$, and energy

$\mathcal{E}_{x_1} = \sum_{n=-\infty}^{\infty}|x_1[n]|^2 = \sum_{n=-\infty}^{\infty} 1 = \infty$.

(b) $x_2[n] = \mu[n]$. Then average power

$\mathcal{P}_{x_2} = \lim_{K\to\infty}\frac{1}{2K+1}\sum_{n=-K}^{K}|x_2[n]|^2 = \lim_{K\to\infty}\frac{1}{2K+1}\sum_{n=0}^{K}1 = \lim_{K\to\infty}\frac{K+1}{2K+1} = \frac{1}{2}$, and energy

$\mathcal{E}_{x_2} = \sum_{n=-\infty}^{\infty}|x_2[n]|^2 = \sum_{n=0}^{\infty}1 = \infty$.

(c) $x_3[n] = n\mu[n]$. Then average power

$\mathcal{P}_{x_3} = \lim_{K\to\infty}\frac{1}{2K+1}\sum_{n=-K}^{K}|x_3[n]|^2 = \lim_{K\to\infty}\frac{1}{2K+1}\sum_{n=1}^{K}n^2 = \lim_{K\to\infty}\frac{K(K+1)(2K+1)}{6} = \infty$,

and energy $\mathcal{E}_{x_3} = \sum_{n=-\infty}^{\infty}|x_3[n]|^2 = \sum_{n=0}^{\infty}n^2 = \infty$.

(d) $x_4[n] = A_0 e^{j\omega_0 n}$. Then average power $P_{x_4} = \lim_{K\to\infty}\frac{1}{2K+1}\sum_{n=-K}^{K}|x_4[n]|^2$

$$= \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=-K}^{K} \left|A_0 e^{j\omega_0 n}\right|^2 = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=-K}^{K} A_0^2 = \lim_{K\to\infty} \frac{1}{2K+1} \cdot A_0^2 (2K+1) = A_0^2,$$

and energy $\mathcal{E}_{x_3} = \sum_{n=-\infty}^{\infty} |x_3[n]|^2 = \sum_{n=-\infty}^{\infty} \left|A_0 e^{j\omega_0 n}\right|^2 = \sum_{n=-\infty}^{\infty} A_0^2 = \infty.$

(e) $x_5[n] = A\cos\left(\frac{2\pi n}{M} + \phi\right)$. Note $x_5[n]$ is a periodic sequence. Then average power

$$P_{x_5} = \frac{1}{M} \sum_{n=0}^{M-1} |x_5[n]|^2 = \frac{1}{M} \sum_{n=0}^{M-1} \left|A\cos\left(\frac{2\pi n}{M} + \phi\right)\right|^2 = \frac{1}{M} \cdot \frac{A^2}{2} \sum_{n=0}^{M-1} \left(\cos\left(\frac{4\pi n}{M} + 2\phi\right) + 1\right).$$

Let $C = \sum_{n=0}^{M-1} \cos\left(\frac{4\pi n}{M} + 2\phi\right)$ and $S = \sum_{n=0}^{M-1} \cos\left(\frac{4\pi n}{M} + 2\phi\right)$. Then

$$C + jS = \sum_{n=0}^{M-1} e^{j\left(\frac{4\pi n}{M} + 2\phi\right)} = e^{j2\phi} \sum_{n=0}^{M-1} e^{j4\pi n/M} = e^{j2\phi} \cdot \frac{1 - e^{j4\pi}}{1 - e^{j4\pi/M}} = 0.$$

Hence $C = 0$. Therefore $P_{x_5} = \frac{1}{M} \cdot \frac{A^2}{2} \sum_{n=0}^{M-1} 1 = \frac{A^2}{2}.$

Since $x_5[n]$ is a periodic sequence, it has infinite energy.

2.26 In each of the following parts, N denotes the fundamental period and r is a positive integer.

(a) $\tilde{x}_1[n] = 4\cos(2\pi n/5)$. Here N and r must satisfy the relation $\frac{2\pi}{5} \cdot N = 2\pi r.$

Among all positive solutions for N and r, the smallest values are $N = 5$ and $r = 1$. Hence the average power is given by

$$P_{x_1} = \frac{1}{N} \sum_{n=0}^{N-1} |\tilde{x}_1[n]|^2 = \frac{1}{5} \sum_{n=0}^{4} \left|4\cos\left(\frac{2\pi n}{5}\right)\right|^2 = 8.$$

(b) $\tilde{x}_2[n] = 3\cos(3\pi n/5)$. Here N and r must satisfy the relation $\frac{3\pi}{5} \cdot N = 2\pi r.$

Among all positive solutions for N and r, the smallest values are $N = 10$ and $r = 3$. Hence the average power is given by

$$P_{x_2} = \frac{1}{N} \sum_{n=0}^{N-1} |\tilde{x}_2[n]|^2 = \frac{1}{10} \sum_{n=0}^{9} \left|3\cos\left(\frac{3\pi n}{5}\right)\right|^2 = 4.5.$$

(c) $\tilde{x}_3[n] = 2\cos(3\pi n/7)$. Here N and r must satisfy the relation $\frac{3\pi}{7} \cdot N = 2\pi r.$

Among all positive solutions for N and r, the smallest values are $N = 14$ and $r = 3$. Hence the average power is given by

$$P_{x_3} = \frac{1}{N} \sum_{n=0}^{N-1} |\tilde{x}_3[n]|^2 = \frac{1}{14} \sum_{n=0}^{13} \left|2\cos\left(\frac{3\pi n}{7}\right)\right|^2 = 2.$$

(d) $\tilde{x}_4[n] = 4\cos(5\pi n/3)$. Here N and r must satisfy the relation $\frac{5\pi}{3}\cdot N = 2\pi r$. Among all positive solutions for N and r, the smallest values are $N = 6$ and $r = 5$. Hence the average power is given by

$$P_{x_4} = \frac{1}{N}\sum_{n=0}^{N-1}\left|\tilde{x}_4[n]\right|^2 = \frac{1}{6}\sum_{n=0}^{5}\left|4\cos\left(\frac{5\pi n}{3}\right)\right|^2 = 8.$$

(e) $\tilde{x}_5[n] = 4\cos(2\pi n/5) + 3\cos(3\pi n/5)$. We first determine the fundamental period N_1 of $\cos(2\pi n/5)$. Here N_1 and r must satisfy the relation $\frac{2\pi}{5}\cdot N_1 = 2\pi r$. Among all positive solutions for N_1 and r, the smallest values are $N_1 = 5$ and $r = 1$. We next determine the fundamental period N_2 of $\cos(3\pi n/5)$. Here N_2 and r must satisfy the relation $\frac{3\pi}{5}\cdot N_2 = 2\pi r$. Among all positive solutions for N_2 and r, the smallest values are $N_2 = 10$ and $r = 3$. The fundamental period of $\tilde{x}_5[n]$ is then given by $LCM(N_1, N_2) = LCM(5,10) = 10.$

Hence the average power is given by

$$P_{x_5} = \frac{1}{N}\sum_{n=0}^{N-1}\left|\tilde{x}_5[n]\right|^2 = \frac{1}{10}\sum_{n=0}^{4}\left|4\cos\left(\frac{2\pi n}{5}\right) + 3\cos\left(\frac{3\pi n}{5}\right)\right|^2$$

$$= \frac{1}{10}\left(\sum_{n=0}^{11}16\cos^2\left(\frac{2\pi n}{5}\right) + \sum_{n=0}^{11}9\cos^2\left(\frac{3\pi n}{5}\right) + \sum_{n=0}^{11}24\left|\cos\left(\frac{2\pi n}{5}\right)\cos\left(\frac{3\pi n}{5}\right)\right|\right) \cong 8 + 4.5 + 0 = 12.5.$$

(f) $\tilde{x}_6[n] = 4\cos(5\pi n/3) + 3\cos(3\pi n/5)$. We first determine the fundamental period N_1 of $\cos(5\pi n/3)$. Here N_1 and r must satisfy the relation $\frac{5\pi}{3}\cdot N_1 = 2\pi r$. Among all positive solutions for N_1 and r, the smallest values are $N_1 = 6$ and $r = 5$. We next determine the fundamental period N_2 of $\cos(3\pi n/5)$. Here N_2 and r must satisfy the relation $\frac{3\pi}{5}\cdot N_2 = 2\pi r$. Among all positive solutions for N_2 and r, the smallest values are $N_2 = 10$ and $r = 3$. The fundamental period of $\tilde{x}_6[n]$ is then given by $LCM(N_1, N_2) = LCM(6,10) = 30.$

Hence the average power is given by

$$P_{x_6} = \frac{1}{N}\sum_{n=0}^{N-1}\left|\tilde{x}_6[n]\right|^2 = \frac{1}{30}\sum_{n=0}^{29}\left|4\cos\left(\frac{5\pi n}{3}\right) + 3\cos\left(\frac{3\pi n}{5}\right)\right|^2$$

$$= \frac{1}{30}\left(\sum_{n=0}^{29}16\cos^2\left(\frac{5\pi n}{3}\right) + \sum_{n=0}^{30}9\cos^2\left(\frac{3\pi n}{5}\right) + \sum_{n=0}^{29}24\left|\cos\left(\frac{5\pi n}{3}\right)\cos\left(\frac{3\pi n}{5}\right)\right|\right) \cong 8 + 4.5 + 0 = 12.5.$$

2.27 Now, from Eq. (2.38) we have $\tilde{y}[n] = \sum_{k=-\infty}^{\infty} x[n+kN]$. Therefore

$\tilde{y}[n+N] = \sum_{k=-\infty}^{\infty} x[n+kN+N]$. Substituting $r = k+1$ we get

$\tilde{y}[n+N] = \sum_{r=-\infty}^{\infty} x[n+rN] = \tilde{y}[n]$. Hence $\tilde{y}[n]$ is a periodic sequence with a period N.

2.28 **(a)** $N = 5$. Now $\tilde{x}_p[n] = \sum_{n=-\infty}^{\infty} x[n+k5]$. The portion of $\tilde{x}_p[n]$ in the range $0 \le n \le 4$ is given by $x[n-5]+x[n]+x[n+5] = \{0 \quad 0 \quad -4 \quad 5 \quad 1\}$
$+\{-2 \quad -3 \quad 0 \quad 2 \quad 0\}+\{0 \quad 0 \quad 0 \quad 0 \quad 0\} = \{-2 \quad -3 \quad -4 \quad 7 \quad 1\}, 0 \le n \le 4$.
Hence, one period of $\tilde{x}_p[n]$ is given by $\{-2 \quad -3 \quad -4 \quad 7 \quad 1\}, 0 \le n \le 4$.

Now $\tilde{y}_p[n] = \sum_{n=-\infty}^{\infty} y[n+k5]$. The portion of $\tilde{y}_p[n]$ in the range $0 \le n \le 4$ is given by
$y[n-5]+y[n]+y[n+5] = \{0 \quad 0 \quad 0 \quad 0 \quad 6\}$
$+\{-3 \quad -1 \quad 0 \quad 8 \quad 7\}+\{-2 \quad 0 \quad 0 \quad 0 \quad 0\} = \{-5 \quad -1 \quad 0 \quad 8 \quad 13\}, 0 \le n \le 4$.
Hence, one period of $\tilde{y}_p[n]$ is given by $\{-5 \quad -1 \quad 0 \quad 8 \quad 13\}, 0 \le n \le 4$.

Now $\tilde{w}_p[n] = \sum_{n=-\infty}^{\infty} w[n+k5]$. The portion of $\tilde{w}_p[n]$ in the range $0 \le n \le 4$ is given by
$w[n-5]+w[n]+w[n+5] = \{0 \quad 0 \quad 0 \quad 0 \quad 0\}$
$+\{0 \quad 0 \quad 3 \quad 2 \quad 2\}+\{-1 \quad 0 \quad -2 \quad 5 \quad 0\} = \{-1 \quad 0 \quad 1 \quad 7 \quad 2\}, 0 \le n \le 4$.
Hence, one period of $\tilde{w}_p[n]$ is given by $\{-1 \quad 0 \quad 1 \quad 7 \quad 2\}, 0 \le n \le 4$.

(b) $N = 7$. Now $\tilde{x}_p[n] = \sum_{n=-\infty}^{\infty} x[n+k7]$. The portion of $\tilde{x}_p[n]$ in the range $0 \le n \le 6$ is given by $x[n-7]+x[n]+x[n+7] = \{0 \quad 0 \quad 0 \quad 0 \quad -4 \quad 5 \quad 1\}$
$+\{-2 \quad -3 \quad 0 \quad 2 \quad 0 \quad 0 \quad 0\}+\{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\}$
$= \{-2 \quad -3 \quad 0 \quad 2 \quad -4 \quad 5 \quad 1\}, 0 \le n \le 6$. Hence, one period of $\tilde{x}_p[n]$ is given by
$\{-2 \quad -3 \quad 0 \quad 2 \quad -4 \quad 5 \quad 1\}, 0 \le n \le 6$.

Now $\tilde{y}_p[n] = \sum_{n=-\infty}^{\infty} y[n+k7]$. The portion of $\tilde{y}_p[n]$ in the range $0 \le n \le 6$ is given by
$x[n-7]+x[n]+x[n+7] = \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 6\}$
$+\{-3 \quad -1 \quad 0 \quad 8 \quad 7 \quad -2 \quad 0\}+\{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\}$
$= \{-3 \quad -1 \quad 0 \quad 8 \quad 7 \quad -2 \quad 6\}, 0 \le n \le 6$. Hence, one period of $\tilde{y}_p[n]$ is given by
$\{-3 \quad -1 \quad 0 \quad 8 \quad 7 \quad -2 \quad 6\}, 0 \le n \le 6$.

Now $\tilde{w}_p[n] = \sum_{n=-\infty}^{\infty} w[n+k7]$. The portion of $\tilde{w}_p[n]$ in the range $0 \le n \le 6$ is given by
$w[n-7]+w[n]+w[n+7] = \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\}$
$+\{0 \quad 0 \quad 3 \quad 2 \quad 2 \quad -1 \quad 0\}+\{-2 \quad 5 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\}$

$=\{-2 \quad 5 \quad 3 \quad 2 \quad 2 \quad -1 \quad 0\}, 0 \le n \le 6.$ Hence, one period of $\tilde{w}_p[n]$ is given by $\{-2 \quad 5 \quad 3 \quad 2 \quad 2 \quad -1 \quad 0\}, 0 \le n \le 6.$

2.29 $\tilde{x}[n] = A\cos(\omega_o n + \phi).$

(a) $\tilde{x}[n] = \{1 \quad -1 \quad -1 \quad 1 \quad 1 \quad -1 \quad -1 \quad 1\}.$ Hence $A = \sqrt{2}, \omega_o = \pi/2, \phi = \pi/4.$

(b) $\tilde{x}[n] = \{0 \quad -\sqrt{3} \quad 0 \quad \sqrt{3} \quad 0 \quad -\sqrt{3} \quad 0 \quad \sqrt{3}\}.$ Hence $A = \sqrt{3},\ \omega_o = \pi/2,$ $\phi = \pi/2.$

(c) $\tilde{x}[n] = \{1 \quad -0.366 \quad -1.366 \quad -1 \quad 0.366 \quad 1.366\}.$ Hence $A = \sqrt{2},\ \omega_o = \pi/3,$ $\phi = \pi/4.$

(d) $\tilde{x}[n] = \{2 \quad 0 \quad -2 \quad 0 \quad 2 \quad 0 \quad -2 \quad 0\}.$ Hence $A = 2, \omega_o = \pi/2, \phi = 0.$

2.30 The fundamental period N of a periodic sequence with an angular frequency ω_o satisfies Eq. (2.47a) with the smallest value of N and r.

(a) $\omega_o = 0.5\pi.$ Here Eq. (2.47a) reduces to $0.5\pi N = 2\pi r$ which is satisfied with $N = 4, r = 1.$

(b) $\omega_o = 0.8\pi.$ Here Eq. (2.47a) reduces to $0.8\pi N = 2\pi r$ which is satisfied with $N = 5, r = 2.$

(c) We first determine the fundamental period N_1 of $\text{Re}\{e^{j\pi n/5}\} = \cos(0.2\pi n).$ In this case, Eq. (2.47a) reduces to $0.2\pi N_1 = 2\pi r_1$ which is satisfied with $N_1 = 10, r_1 = 1.$ We next determine the fundamental period N_2 of $\text{Im}\{e^{j\pi n/10} = j\sin(0.1\pi n).$ In this case, Eq. (2.47a) reduces to $0.1\pi N_2 = 2\pi r_2$ which is satisfied with $N_2 = 20, r_2 = 1.$ Hence the fundamental period N of $\tilde{x}_c[n]$ is given by $LCM(N_1, N_2) = LCM(10,20) = 20.$

(d) We first determine the fundamental period N_1 of $3\cos(1.3\pi n).$ In this case, Eq. (2.47a) reduces to $1.3\pi N_1 = 2\pi r_1$ which is satisfied with $N_1 = 20, r_1 = 13.$ We next determine the fundamental period N_2 of $4\sin(0.5\pi n + 0.5\pi).$ In this case, Eq. (2.47a) reduces to $0.5\pi N_2 = 2\pi r_2$ which is satisfied with $N_2 = 4, r_2 = 1.$ Hence the fundamental period N of $\tilde{x}_4[n]$ is given by $LCM(N_1, N_2) = LCM(20,4) = 20.$

(e) We first determine the fundamental period N_1 of $5\cos(1.5\pi n + 0.75\pi).$ In this case, Eq. (2.47a) reduces to $1.5\pi N_1 = 2\pi r_1$ which is satisfied with $N_1 = 4, r_1 = 3.$ We next determine the fundamental period N_2 of $4\cos(0.6\pi n).$ In this case, Eq. (2.47a) reduces to $0.6\pi N_2 = 2\pi r_2$ which is satisfied with $N_2 = 10, r_2 = 3.$ We finally determine the fundamental period N_3 of $\sin(0.5\pi n).$ In this case, Eq. (2.47a) reduces to $0.5\pi N_3 = 2\pi r_3$ which is satisfied with $N_3 = 4, r_3 = 1.$ Hence the fundamental period N of $\tilde{x}_5[n]$ is given by $LCM(N_1, N_2, N_3) = LCM(4,10,4) = 20.$

2.31 The fundamental period N of a periodic sequence with an angular frequency ω_o satisfies Eq. (2.47a) with the smallest value of N and r.

(a) $\omega_o = 0.6\pi.$ Here Eq. (2.47a) reduces to $0.6\pi N = 2\pi r$ which is satisfied with $N = 10, r = 3.$

(b) $\omega_o = 0.28\pi.$ Here Eq. (2.47a) reduces to $0.28\pi N = 2\pi r$ which is satisfied with $N = 50, r = 7.$

(c) $\omega_o = 0.45\pi.$ Here Eq. (2.47a) reduces to $0.45\pi N = 2\pi r$ which is satisfied with $N = 40, r = 9.$

(d) $\omega_o = 0.55\pi.$ Here Eq. (2.47a) reduces to $0.55\pi N = 2\pi r$ which is satisfied with $N = 40, r = 11.$

(e) $\omega_o = 0.65\pi.$ Here Eq. (2.47a) reduces to $0.65\pi N = 2\pi r$ which is satisfied with $N = 40, r = 13.$

2.32 $\omega_o = 0.08\pi.$ Here Eq. (2.47a) reduces to $0.08\pi N = 2\pi r$ which is satisfied with $N = 25, r = 1.$ For a sequence $\tilde{x}_2[n] = \sin(\omega_2 n)$ with a fundamental period of $N = 25$, Eq. (2.47a) reduces to $25\omega_2 = 2\pi r.$ For example, for $r = 2$ we have $\omega_2 = 4\pi/25 = 0.16\pi.$ Another sequence with the same fundamental period is obtained by setting $r = 3$ which leads to $\omega_3 = 6\pi/25 = 0.24\pi.$ The corresponding periodic sequences are therefore $\tilde{x}_2[n] = \sin(0.16\pi n)$ and $\tilde{x}_3[n] = \sin(0.24\pi n).$

2.33 The three parameters $A, \Omega_o,$ and ϕ of the continuous-time signal $x_a(t)$ can be determined from $x[n] = x_a(nT) = A\cos(\Omega_o nT + \phi)$ by setting 3 distinct values of n. For example

$$x[0] = A\cos\phi = \alpha,$$
$$x[-1] = A\cos(-\Omega_o T + \phi) = A\cos(\Omega_o T)\cos\phi + A\sin(\Omega_o T)\sin\phi = \beta, ,$$
$$x[1] = A\cos(\Omega_o T + \phi) = A\cos(\Omega_o T)\cos\phi - A\sin(\Omega_o T)\sin\phi = \gamma.$$

Substituting the first equation into the last two equations and then adding them we get $\cos(\Omega_o T) = \dfrac{\beta + \gamma}{2\alpha}$ which can be solved to determine Ω_o. Next, from the second equation we have $A\sin\phi = \beta - A\cos(\Omega_o T)\cos\phi = \beta - \alpha\cos(\Omega_o T).$ Dividing this equation by the last equation on the previous page we arrive at $\tan\phi = \dfrac{\beta - \alpha\cos(\Omega_o T)}{\alpha\sin(\Omega_o T}$ which can be solved to determine $\phi.$ Finally, the parameter is determined from the first equation of the last page.

Now consider the case $\Omega_T = \frac{2\pi}{T} = 2\Omega_o$. In this case $x[n] = A\cos(n\pi + \phi) = \beta$ and $x[n+1] = A\cos\big((n+1)\pi + \phi\big) = A\cos(n\pi + \phi) = \beta$. Since all sample values are equal, the three parameters cannot be determined uniquely.

Finally consider the case $\Omega_T = \frac{2\pi}{T} < 2\Omega_o$. In this case $x[n] = A\cos(\Omega_o nT + \phi)$ $= A\cos(\omega_o n + \phi)$ implying $\omega_o = \Omega_o T > \pi$. As explained in Section 2.2.1, a digital sinusoidal sequence with an angular frequency ω_o greater than π assumes the identity of a sinusoidal sequence with an angular frequency in the range $0 \le \omega < \pi.$. Hence, Ω_o cannot be uniquely determined from $x[n] = A\cos(\Omega_o nT + \phi)$.

2.34 $x[n] = \cos(\Omega_o nT)$. If $x[n]$ is periodic with a period N, then $x[n+N] = \cos\big(\Omega_0 nT + \Omega_0 NT\big) = x[n] = \cos(\Omega_0 nT)$. This implies $\Omega_o NT = 2\pi r$ with r any nonzero positive integer. Hence the sampling rate must satisfy the relation $T = 2\pi r / \Omega_o N$. If $\Omega_o = 20$, i.e., $T = \pi/8$, then we must have $20N \cdot \frac{\pi}{8} = 2\pi r$. The smallest value of N and r satisfying this relation are $N = 4$ and $r = 5$. The fundamental period is thus $N = 4$.

2.35 **(a)** For an input $x_i[n], i = 1,2,$ the output is $y_i[n] = b_0 x_i[n] + b_1 x_i[n-1] + b_2 x_i[n-2] + a_1 y_i[n-1] + a_2 y_i[n-2], i = 1,2.$ Then, for an input $x[n] = Ax_1[n] + Bx_2[n]$, the output is $y[n] = b_0(Ax_1[n] + Bx_2[n])$ $+ b_1(Ax_1[n-1] + Bx_2[n-1]) + b_2(Ax_1[n-2] + Bx_2[n-2]) + a_1(Ay_1[n-1] + By_2[n-1])$ $+ a_1(Ay_1[n-2] + By_2[n-2]) = A(b_0 x_1[n] + b_1 x_1[n-1] + b_2 x_1[n-2] + a_1 y_1[n-1]$ $+ a_2 y_2[n-1]) + B(b_1 x_2[n] + b_2 x_2[n-1] + b_3 x_2[n-2] + a_1 x_2[n-1] + a_2 x_2[n-2])$ $= Ay_1[n] + By_2[n].$ Hence, the system of Eq. (2.18) is linear.

(b) For an input $x_i[n], i = 1,2,$ the output is $y_i[n] = \begin{cases} x_i[n/L], & n = 0, \pm L, \pm 2L, \cdots \\ 0, & \text{otherwise.} \end{cases}$

For an input $x[n] = Ax_1[n] + Bx_2[n]$, the output for $n = 0, \pm L, \pm 2L, \ldots$ is $y[n] = x[n/L] = Ax_1[n/L] + Bx_2[n/L] = Ay_1[n] + By_2[n]$. For all other values of $n, y[n] = A \cdot 0 + B \cdot 0 = 0.$ Hence the system of Eq. (2.20) is linear.

(c) For an input $x_i[n], i = 1,2,$ the output is $y_i[n] = x_i[n/M], i = 1,2.$ Then, for an input $x[n] = Ax_1[n] + Bx_2[n]$, the output is $y[n] = Ax_1[n/M] + Bx_2[n/M] = Ay_1[n] + By_2[n].$ Hence the system of Eq. (2.21) is linear.

(d) For an input $x_i[n], i=1,2,$ the output is $y_i[n]=\frac{1}{M}\sum_{k=0}^{M-1} x_i[n-k], i=1,2.$ Then, for an input $x[n]=Ax_1[n]+Bx_2[n],$ the output is $y[n]=\frac{1}{M}\sum_{k=0}^{M-1}\left(Ax_1[n-k]+Bx_2[n-k]\right)$ $=A\left(\frac{1}{M}\sum_{k=0}^{M-1}x_1[n-k]\right)+B\left(\frac{1}{M}\sum_{k=0}^{M-1}x_2[n-k]\right)=Ay_1[n]+By_2[n].$ Hence the system of Eq. (2.61) is linear.

(e) The first term on the RHS of Eq. (2.65) is the output of a factor-of-2 up-sampler. The second term on the RHS of Eq. (2.65) is simply the output of an unit delay followed by a factor-of-2 up-sampler, whereas, the third term is the output of an unit advance operator followed by a factor-of-2 up-sampler. We have shown in Part (b) that the up-sampler is a linear system. Moreover, the unit delay and the unit advance operator are linear systems. A cascade of two linear systems is linear and the linear combination of linear systems is also linear. Hence, the factor-of-2 interpolator of Eq. (2.65) is a linear system.

(f) Following the arguments given in Part (e), we can similarly show that the factor-of-3 interpolator of Eq. (2.66) is a linear system.

2.36 **(a)** $y[n]=n^3x[n].$ For an input $x_i[n], i=1,2,$ the output is $y_i[n]=n^3x_i[n], i=1,2.$ Then, for an input $x[n]=Ax_1[n]+Bx_2[n],$ the output is $y[n]=n^3\left(Ax_1[n]+Bx_2[n]\right)$ $=Ay_1[n]+By_2[n].$ Hence the system is linear.

For an input $x[n]=\delta[n],$ the output is the impulse response $h[n]=n^3\delta[n].$ As $h[n]=0$ for $n<0,$ and the system is causal.

Let $x[n]=1$ for all values of $n.$ Then $|y[n]|=|n^3|$ and $y[n]\to\infty$ as $n\to\infty.$ Since a bounded input results in an unbounded output, the system is not BIBO stable.

Finally, let $y[n]$ and $y_1[n]$ be the outputs for inputs $x[n]$ and $x_1[n],$ respectively. If $x_1[n]=x[n-n_o]$ then $y_1[n]=n^3x_1[n]=n^3x[n-n_o].$ However, $y[n-n_o]=$ $(n-n_o)^3x[n-n_o].$ Since $y_1[n]\neq y[n-n_o],$ the system is not time-invariant.

(b) $y[n]=(x[n])^5.$ For an input $x_i[n], i=1,2,$ the output is $y_i[n]=(x_i[n])^5, i=1,2.$ Then, for an input $x[n]=Ax_1[n]+Bx_2[n],$ the output is $y[n]=\left(Ax_1[n]+Bx_2[n]\right)^5$ $\neq A(x_1[n])^5+B(x_2[n])^5.$ Hence the system is nonlinear.

For an input $x[n]=\delta[n],$ the output is the impulse response $h[n]=(\delta[n])^5.$ As $h[n]=0$ for $n<0,$ and the system is causal.

For a bounded input $|x[n]| \le B < \infty,$ the magnitude of the output samples are $|y[n]| = |(x[n])^5| = |x[n]|^5 \le B^5 < \infty.$ As the output is also a bounded sequence, the system is BIBO stable.

Finally, let $y[n]$ and $y_1[n]$ be the outputs for inputs $x[n]$ and $x_1[n],$ respectively. If $x_1[n] = x[n-n_o]$ then $y_1[n] = (x_1[n])^5 = (x[n-n_o])^5 = y[n-n_o].$ Hence, the system is time-invariant.

(c) $y[n] = \beta + \sum_{\ell=0}^{3} x[n-\ell]$ with β a nonzero constant. For an input $x_i[n], i = 1,2,$ the output is $y_i[n] = \beta + \sum_{\ell=0}^{3} x_i[n-\ell], i = 1,2.$ Then, for an input $x[n] = Ax_1[n] + Bx_2[n],$ the output is $y[n] = \beta + \sum_{\ell=0}^{3}(Ax_1[n-\ell] + Bx_2[n-\ell]) = \beta + \sum_{\ell=0}^{3} Ax_1[n-\ell] + \sum_{\ell=0}^{3} Bx_2[n-\ell]$ $\ne Ay_1[n] + By_2[n].$ Hence the system is nonlinear.

For an input $x[n] = \delta[n],$ the output is the impulse response $h[n] = \beta + \sum_{\ell=0}^{\infty} \delta[n-\ell].$ As $h[n] \ne 0$ for $n < 0,$ the system is noncausal.

For a bounded input $|x[n]| \le B < \infty,$ the magnitude of the output samples are $|y[n]| \le \beta + 4B < \infty.$ As the output is also a bounded sequence, the system is BIBO stable.

Finally, let $y[n]$ and $y_1[n]$ be the outputs for inputs $x[n]$ and $x_1[n],$ respectively. If $x_1[n] = x[n-n_o]$ then $y_1[n] = \beta + \sum_{\ell=0}^{3} x[n - n_o - \ell] = y[n-n_o].$ Hence, the system is time-invariant.

(d) $y[n] = \ln(2 + |x[n]|).$ For an input $x_i[n], i = 1,2,$ the output is $y_i[n] = \ln(2 + |x_i[n]|),$ $i = 1,2.$ Then, for an input $x[n] = Ax_1[n] + Bx_2[n],$ the output is $y[n] = \ln(2 + |Ax_1[n] + Bx_2[n]|) \ne Ay_1[n] + By_2[n].$ Hence the system is nonlinear.

For an input $x[n] = \delta[n],$ the output is the impulse response $h[n] + \ln(2 + |\delta[n]|).$

For $n < 0,$ $h[n] = \ln(2) \ne 0.$ Hence, the system is noncausal.

For a bounded input $|x[n]| \le B < \infty,$ the magnitude of the output samples are $|y[n]| \le \ln(2 + B) < \infty.$ As the output is also a bounded sequence, the system is BIBO stable.

Finally, let $y[n]$ and $y_1[n]$ be the outputs for inputs $x[n]$ and $x_1[n],$ respectively. If

$x_1[n] = x[n-n_o]$ then $y_1[n] = \ln(2 + |x[n-n_o]|) = y[n-n_o]$. Hence, the system is time-invariant.

(e) $y[n] = \alpha x[-n+2]$, with a nonzero constant. For an input $x_i[n], i = 1,2,$ the output is $y_i[n] = \alpha x_i[-n+2]$, $i = 1,2$. Then, for an input $x[n] = Ax_1[n] + Bx_2[n]$, the output is $y[n] = A\alpha x_1[-n+2] + B\alpha x_2[-n+2] = Ay_1[n] + By_2[n]$. Hence the system is linear.
For an input $x[n] = \delta[n]$, the output is the impulse response $h[n] = \alpha\delta[-n+2]$. For $n < 0$, $h[n] = 0$. Hence, the system is causal.
For a bounded input $|x[n]| \le B < \infty$, the magnitude of the output samples are $|y[n]| = |\alpha B| < \infty$. As the output is also a bounded sequence, the system is BIBO stable.
Finally, let $y[n]$ and $y_1[n]$ be the outputs for inputs $x[n]$ and $x_1[n]$, respectively. If $x_1[n] = x[n-n_o]$ then $y_1[n] = \alpha x_1[-n+2] = \alpha x[-(n-n_o)+2] = y[n-n_o]$. Hence, the system is time-invariant.

(f) $y[n] = x[n-4]$. For an input $x_i[n], i = 1,2,$ the output is $y_i[n] = x_i[n-4], i = 1,2.$
Then, for an input $x[n] = Ax_1[n] + Bx_2[n]$, the output is $y[n] = Ax_1[n-4] + Bx_2[n-4]$ $= Ay_1[n] + By_2[n]$. Hence the system is linear.
For an input $x[n] = \delta[n]$, the output is the impulse response $h[n] = \delta[n-4]$. For $n < 0$, $h[n] = 0$. Hence, the system is causal.
For a bounded input $|x[n]| \le B < \infty$, the magnitude of the output samples are $|y[n]| = B < \infty$. As the output is also a bounded sequence, the system is BIBO stable.
Finally, let $y[n]$ and $y_1[n]$ be the outputs for inputs $x[n]$ and $x_1[n]$, respectively. If $x_1[n] = x[n-n_o]$ then $y_1[n] = x[n-n_o-4] = y[n-n_o]$. Hence, the system is time-invariant.

2.37 Let $y[n]$ and $y_1[n]$ be the outputs of a median filter of length $2K+1$ for inputs $x[n]$ and $x_1[n]$, respectively. If $x_1[n] = x[n-n_o]$, then
$y_1[n] = \text{med}\{x_1[n-K], \ldots, x_1[n-1], x_1[n], x_1[n+1], \ldots, x_1[n+K]\}$
$= \text{med}\{x[n-n_o-K], \ldots, x[n-n_o-1], x[n-n_o], x[n-n_o+1], \ldots, x[n-n_o+K]\}$
$= y[n-n_o]$. Hence, the system is time-invariant.

2.38 $y[n] = x[n+1] - 2x[n] + x[n-1]$. For an input $x_i[n], i = 1,2,$ the output is $y_i[n] = x_i[n+1] - 2x_i[n] + x_i[n-1], i = 1,2.$ Then, for an input $x[n] = Ax_1[n] + Bx_2[n]$, the output is
$y[n] = Ax_1[n+1] + Bx_2[n+1] - 2Ax_1[n] - 2Bx_2[n] + Ax_1[n-1] + Bx_2[n-1]$
$= Ay_1[n] + By_2[n]$. Hence the system is linear.
If $x_1[n] = x[n-n_o]$, then $y_1[n] = x[n-n_o+1] - 2x[n-n_o] + x[-n_o n-1] = y[n-n_o]$.
Hence, the system is time-invariant.

The impulse response of the system is $h[n] = \delta[n+1] - 2\delta[n] + \delta[n-1]$. Now $h[-1] = \delta[0] = 1$. Since $h[n] \neq 0$ for all values of $n < 0$, the system is noncausal.

2.39 $y[n] = x^2[n] - x[n-1]x[n+1]$. For an input $x_i[n], i = 1,2,$ the output is $y_i[n] = x_i^2[n] - x_i[n-1]x_i[n+1], i = 1,2.$ Then, for an input $x[n] = Ax_1[n] + Bx_2[n]$, the output is $y[n] = \left(Ax_1[n] + Bx_2[n]\right)^2 - \left(Ax_1[n-1] + Bx_2[n-1]\right)\left(Ax_1[n+1] + Bx_2[n+1]\right)$ $\neq Ay_1[n] + By_2[n]$. Hence the system is nonlinear.

If $x_1[n] = x[n-n_o]$, then $y_1[n] = x_1^2[n] - x_1[n-1]x_1[n+1]$ $= x^2[n-n_o] - x[n-n_o-1]x[n-n_o+1] = y[n-n_o]$. Hence, the system is time-invariant.

The impulse response of the system is $h[n] = \delta^2[n] - \delta[n-1]\delta[n+1] = \delta[n]$. Since $h[n] = 0$ for all values of $n < 0$, the system is causal.

2.40 $y[n] = \dfrac{1}{2}\left(y[n-1] + \dfrac{x[n]}{y[n-1]}\right)$. Now for an input $x[n] = \alpha\mu[n]$, the output $y[n]$ converges to some constant K as $n \to \infty$. The input-output relation of the system as $n \to \infty$ reduces to $K = \dfrac{1}{2}\left(K + \dfrac{\alpha}{K}\right)$ from which we get $K^2 = \alpha$ or in other words $K = \sqrt{\alpha}$.

For an input $x_i[n], i = 1,2,$ the output is $y_i[n] = \dfrac{1}{2}\left(y_i[n-1] + \dfrac{x_i[n]}{y_i[n-1]}\right), i = 1,2.$ Then, for an input $x[n] = Ax_1[n] + Bx_2[n]$, the output is $y[n] = \dfrac{1}{2}\left(y[n-1] + \dfrac{Ax_1[n] + Bx_2[n]}{y[n-1]}\right)$.

On the other hand,

$$Ay_1[n] + By_2[n] = \frac{1}{2}\left(Ay_1[n-1] + \frac{Ax_1[n]}{y_1[n-1]}\right) + \frac{1}{2}\left(By_2[n-1] + \frac{Bx_2[n]}{y_2[n-1]}\right) \neq y[n].$$

Hence the system is nonlinear.

If $x_1[n] = x[n-n_o]$, then $y_1[n] = \dfrac{1}{2}\left(y_1[n-1] + \dfrac{x[n-n_o]}{y_1[n-1]}\right) = y[n-n_o]$. Hence, the system is time-invariant.

2.41 $y[n] = x[n] - y^2[n-1] + y[n-1]$.

For an input $x_i[n], i = 1,2,$ the output is $y_i[n] = x_i[n] - y_i^2[n-1] + y_i[n-1], i = 1,2.$ Then, for an input $x[n] = Ax_1[n] + Bx_2[n]$, the output is $y[n] = Ax_1[n] + Bx_2[n] - y^2[n-1] + y[n-1]$. On the other hand, $Ay_1[n] + By_2[n]$

$= Ax_1[n] - Ay_1^2[n-1] + Ay_1[n-1] + Bx_2[n] - By_2^2[n-1] + By_2[n-1] \neq y[n].$ Hence the system is nonlinear.

2.42 The impulse response of the factor-of-3 interpolator of Eq. (2.66) is the output for an input $x_u[n] = \delta[n]$ and is given by

$h[n] = \delta[n] + \frac{2}{3}(\delta[n-1] + \delta[n+1]) + \frac{1}{3}(\delta[n-2] + \delta[n+2])$ or equivalently by

$\{h[n]\} = \left\{\frac{1}{3}, \quad \frac{2}{3}, \quad 1, \quad \frac{2}{3}, \quad \frac{1}{3}\right\}, -2 \leq n \leq 2.$

2.43 The input-output relation of a factor-of-L interpolator is given by

$y[n] = x_u[n] + \sum_{k=1}^{L-1} \frac{L-k}{L}\left(x_u[n-k] + x_u[n+k]\right).$ Its impulse response is the output for an input $x_u[n] = \delta[n]$ and is thus given by $h[n] = \delta[n] + \sum_{k=1}^{L-1} \frac{L-k}{L}\left(\delta[n-k] + \delta[n+k]\right)$

or equivalently by

$\{h[n]\} = \left\{\frac{1}{L}, \quad \frac{2}{L}, \quad \ldots, \quad \frac{L-2}{L}, \quad \frac{L-1}{L}, \quad 1, \quad \frac{L-1}{L}, \quad \frac{L-2}{L}, \quad \ldots, \quad \frac{2}{L}, \quad \frac{1}{L}\right\}, -L+1 \leq n \leq L-1.$

2.44 The impulse response $h[n]$ of a causal discrete-time system satisfies the difference equation $h[n] - a\,h[n-1] = \delta[n].$ Since the system is causal, we have $h[n] = 0$ for $n < 0.$ Evaluating the above difference equation for $n = 0,$ we arrive at $h[0] - ah[-1] = 1$ and thus $h[0] = 1.$ Next, for $n = 1$, we have $h[1] - a\,h[0] = 0$ and thus $h[1] = a.$ Continuing we get for $n = 2, h[2] - ah[1] = 0$, i.e., $h[2] = ah[1] = a^2.$ Assume $h[n-1] = a^{n-1}$ with $n > 0.$ From the difference equation we then have $h[n] - a\,h[n-1] = 0$, i.e., $h[n] = ah[n-1] = a^n.$ Since the last equation holds for $n = 0, 1, 2,$ by induction, it holds for $n \geq 3.$

2.45 As $x[n]$ and $h[n]$ are right-sided sequences, assume $x[n] = 0$ for all $n < N_1$ and $h[n] = 0$ and $n < N_2.$ Hence, $y[n] = h[n] \circledast x[n] = 0$ for all $n < N_1 + N_2$ and thus

$y[n]$ is also a right-sided sequence. Therefore, $\sum_{n=N_1+N_2}^{\infty} y[n] = \sum_{n=N_1+N_2}^{\infty} h[n] \circledast x[n]$

$= \sum_{n=N_1+N_2}^{\infty} \sum_{k=N_2}^{\infty} h[k]x[n-k] = \sum_{k=N_2}^{\infty} \sum_{n=N_1+N_2}^{\infty} h[k]x[n-k] = \sum_{k=N_2}^{\infty} h[k] \sum_{n=N_1+N_2}^{\infty} x[n-k]$

$= \sum_{k=N_2}^{\infty} h[k] \sum_{m=N_1+N_2-k}^{\infty} x[m] = \sum_{k=N_2}^{\infty} h[k] \sum_{m=N_1}^{\infty} x[m]$ as $x[m] = 0$ for all $m < N_1.$ Hence,

$\sum_n y[n] = \left(\sum_n h[n]\right)\left(\sum_n x[n]\right).$

2.46 **(a)** $\alpha^n\mu[n] \circledast \mu[n] = \sum_{k=-\infty}^{\infty}\alpha^k\mu[k]\mu[n-k] = \sum_{k=0}^{\infty}\alpha^k\mu[n-k] = \begin{cases}\sum_{k=0}^{n}\alpha^k, & n \geq 0,\\ 0, & n < 0.\end{cases}$

(b) $n\alpha^n\mu[n] \circledast \mu[n] = \sum_{k=-\infty}^{\infty}k\alpha^k\mu[k]\mu[n-k] = \sum_{k=0}^{\infty}k\alpha^k\mu[n-k] = \begin{cases}\sum_{k=0}^{n}k\alpha^k, & n > 0,\\ 0, & n \leq 0.\end{cases}$

2.47 Now from Eq. (2.72) an arbitrary input $x[n]$ can be expressed as

$x[n] = \sum_{k=-\infty}^{\infty}x[k]\delta[n-k]$ which can be rewritten using Eq. (2.41b) as

$$x[n] = \sum_{k=-\infty}^{\infty}x[k]\big(\mu[n-k]-\mu[n-k-1]\big) = \sum_{k=-\infty}^{\infty}x[k]\mu[n-k] - \sum_{k=-\infty}^{\infty}x[k]\mu[n-k-1].$$

Since $s[n]$ is the response of an LTI system for an input $\mu[n]$, $s[n-k]$ is the response for an input $\mu[n-k]$ and $s[n-k-1]$ is the response for an input $\mu[n-k-1]$. Hence, the output for an input $\sum_{k=-\infty}^{\infty}x[k]\mu[n-k] - \sum_{k=-\infty}^{\infty}x[k]\mu[n-k-1]$ is given by

$$y[n] = \sum_{k=-\infty}^{\infty}x[k]s[n-k] - \sum_{k=-\infty}^{\infty}x[k]s[n-k-1] = x[n] \circledast s[n] - x[n-1] \circledast s[n-1].$$

2.48 $y[n] = \sum_{m=-\infty}^{\infty}h[m]\tilde{x}[n-m]$. Hence,

$y[n+kN] = \sum_{m=-\infty}^{\infty}h[m]\tilde{x}[n+kN-m] = \sum_{m=-\infty}^{\infty}h[m]x[n-m] = y[n]$. Thus, $y[n]$ is also a periodic sequence with a period N.

2.49 In this problem we make use of the identity $\delta[n-m] \circledast \delta[n-r] = \delta[n-m-r]$.

(a) $y_1[n] = x_1[n] \circledast h_1[n] = \big(3\delta[n-2]-2\delta[n+1]\big) \circledast \big(-\delta[n+2]+4\delta[n]+2\delta[n-1]\big)$

$= -3\delta[n-2] \circledast \delta[n+2] + 12\delta[n-2] \circledast \delta[n] - 6\delta[n-2] \circledast \delta[n-1] + 2\delta[n+1] \circledast \delta[n+2]$

$-8\delta[n+1] \circledast \delta[n] + 4\delta[n+1] \circledast \delta[n-1]$. Hence

$y_1[n] = -3\delta[n] + 12\delta[n-2] - 6\delta[n-3] + 2\delta[n+3] - 8\delta[n+1] + 4\delta[n]$
$= 2\delta[n+3] - 8\delta[n+1] + \delta[n] + 12\delta[n-2] - 6\delta[n-3]$.

(b) $y_2[n] = x_2[n] \circledast h_2[n] = \big(5\delta[n-3]+2\delta[n+1]\big) \circledast \big(3\delta[n-4]+1.5\delta[n-2]-\delta[n+1]\big)$

$= 15\delta[n-3] \circledast \delta[n-4] + 7.5\delta[n-3] \circledast \delta[n-2] - 5\delta[n-3] \circledast \delta[n+1] + 6\delta[n+1] \circledast \delta[n-4]$

$+3\delta[n+1] \circledast \delta[n-2] - 2\delta[n+1] \circledast \delta[n+1] = 15\delta[n-7] + 7.5\delta[n-5] - 5\delta[n-2]$

$+6\delta[n-3] + 3\delta[n-1] - 2\delta[n+2]$.

(c) $y_3[n] = x_1[n] \circledast h_2[n] = \left(-3\delta[n-2] - 2\delta[n+1]\right) \circledast \left(3\delta[n-4] + 1.5\delta[n-2] - \delta[n+1]\right)$

$= 9\delta[n-2] \circledast \delta[n-4] + 4.5\delta[n-2] \circledast \delta[n-2] - 3\delta[n-2] \circledast \delta[n+1] - 6\delta[n+1] \circledast \delta[n-4]$

$-3\delta[n+1] \circledast \delta[n-2] + 2\delta[n+1] \circledast \delta[n+1] = 9\delta[n-6] + 4.5\delta[n-4] - 3\delta[n-1]$

$-6\delta[n-3] - 3\delta[n-1] - 3\delta[n-1] + 2\delta[n+2] = 2\delta[n+2] - 6\delta[n-1] - 6\delta[n-3]$
$+4.5\delta[n-4] + 9\delta[n-6].$

(d) $y_4[n] = x_2[n] \circledast h_1[n] = \left(5\delta[n-3] + 2\delta[n+1]\right) \circledast \left(-\delta[n+2] + 4\delta[n] - 2\delta[n-1]\right)$

$= -5\delta[n-3] \circledast \delta[n+2] + 20\delta[n-3] \circledast \delta[n] - 10\delta[n-3] \circledast \delta[n-1] - 2\delta[n+1] \circledast \delta[n+2]$

$+8\delta[n+1] \circledast \delta[n] - 4\delta[n+1] \circledast \delta[n-1] = -5\delta[n-1] + 20\delta[n-3] - 10\delta[n-4] - 2\delta[n+3]$

$+8\delta[n+1] - 4\delta[n] = -2\delta[n+3] + 8\delta[n+1] - 4\delta[n] - 5\delta[n-1] + 20\delta[n-3] - 10\delta[n-4].$

2.50 **(a)** $u[n] = x[n] \circledast y[n]$

$= \{-24,\ 42,\ -5,\ -20,\ -45,\ 23,\ 66,\ -25,\ -42,\ -17,\ 22,\ 14,\ -4\},\ -4 \le n \le 8.$

(b) $v[n] = x[n] \circledast w[n]$

$= \{-12,\ 7,\ 5,\ 10,\ -16,\ -3,\ -28,\ 30,\ 13,\ -6,\ -15,\ -4,\ 10\},\ -1 \le n \le 11.$

(c) $g[n] = w[n] \circledast y[n]$

$= \{18,\ 3,\ 3,\ -14,\ 25,\ 26,\ 60,\ -11,\ -16,\ -14,\ 26,\ 39,\ -10\},\ 1 \le n \le 13.$

2.51 $y[n] = \sum_{m=N_1}^{N_2} g[m]h[n-m]$. Now, $h[n-m]$ is defined for $M_1 \le n-m \le M_2$. Thus, for $m = N_1$, $h[n-m]$ is defined for $M_1 \le n - N_1 \le M_2$, or equivalently, for $M_1 + N_1 \le n \le M_2 + N_1$. Likewise, for $m = N_2$, $h[n-m]$ is defined for $M_1 \le n - N_2 \le M_2$, or equivalently, for $M_1 + N_2 \le n \le M_2 + N_2$. For the specified sequences $N_1 = -3, N_2 = 4, M_1 = 2, M_2 = 6$. **(a)** The length of $y[n]$ is $M_2 + N_2 - M_1 - N_1 + 1 = 6 + 4 - 2 - (-3) + 1 = 12$. **(b)** The range of n for $y[n] \ne 0$ is $\min(M_1 + N_1, M_2 + N_2) \le n \le \max(M_1 + N_1, M_2 + N_2)$, i.e., $M_1 + N_1 \le n \le M_2 + N_2$. For the specified sequences the range of n is $-1 \le n \le 10$.

2.52 $y[n] = x_1[n] \circledast x_2[n] = \sum_{k=-\infty}^{\infty} x_1[n-k]x_2[k]$. Now,

$\mathbf{v}[n] = x_1[n-N_1] \circledast x_2[n-N_2] = \sum_{k=-\infty}^{\infty} x_1[n-N_1-k]x_2[k-N_2]$. Let $k - N_2 = m.$

Then $\mathbf{v}[n] = \sum_{m=-\infty}^{\infty} x_1[n-N_1-N_2-m]x_2[m] = y[n-N_1-N_2]$.

2.53 $g[n] = x_1[n] \circledast x_2[n] \circledast x_3[n] = y[n] \circledast x_3[n]$ where $y[n] = x_1[n] \circledast x_2[n]$. Now $v[n] = x_1[n-N_1] \circledast x_2[n-N_2]$. Define $h[n] = v[n] \circledast x_3[n-N_3]$. Then from the results of Problem 2.52, $v[n] = y[n-N_1-N_2]$. Hence, $h[n] = y[n-N_1-N_2] \circledast x_3[n-N_3]$. Therefore, making use of the results of Problem 2.52 again we get $h[n] = y[n-N_1-N_2-N_3]$.

2.54 $y[n] = x[n] \circledast h[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$. Substituting k by $n-m$ in this expression, we get $y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = h[n] \circledast x[n]$. Hence the convolution operation is commutative.

Let $y[n] = x[n] \circledast \left(h_1[n] + h_2[n]\right) = \sum_{k=-\infty}^{\infty} x[n-k]\left(h_1[k] + h_2[k]\right)$

$= \sum_{k=-\infty}^{\infty} x[n-k]h_1[k] + \sum_{k=-\infty}^{\infty} x[n-k]h_2[k] = x[n] \circledast h_1[n] + x[n] \circledast h_2[n]$. Hence the convolution operation is also distributive.

2.55 $x_3[n] \circledast x_2[n] \circledast x_1[n] = x_3[n] \circledast (x_2[n] \circledast x_1[n])$. As $x_2[n] \circledast x_1[n]$ is an unbounded sequence, the result of this convolution cannot be determined. But $x_2[n] \circledast x_3[n] \circledast x_1[n] = x_2[n] \circledast (x_3[n] \circledast x_1[n])$. Now $x_3[n] \circledast x_1[n] = 0$ for all values of n, and hence the overall result is zero. As a result, for the given sequences $x_3[n] \circledast x_2[n] \circledast x_1[n] \neq x_2[n] \circledast x_3[n] \circledast x_1[n]$

2.56 $w[n] = x[n] \circledast h[n] \circledast g[n]$. Define $y[n] = x[n] \circledast h[n] = \sum_k x[k]h[n-k]$ and $f[n] = h[n] \circledast g[n] = \sum_k g[k]h[n-k]$. Consider $w_1[n] = (x[n] \circledast h[n]) \circledast g[n]$ $= y[n] \circledast g[n] = \sum_m g[m] \sum_k x[k]h[n-m-k]$. Now consider $w_2[n] = x[n] \circledast (h[n] \circledast g[n])$

$= x[n] \circledast f[n] = \sum_k x[k] \sum_m g[m]h[n-k-m]$. The difference between the expressions for $w_1[n]$ and $w_2[n]$ is that the order of the summations is changed.

A) Assumptions: $h[n]$ and $g[n]$ are causal sequences, and $x[n]=0$ for $n<0$. This implies $y[m] = \begin{cases} 0, & \text{for } m<0, \\ \sum_{k=0}^{m} x[k]h[m-k], & \text{for } m \geq 0. \end{cases}$ Thus, $w[n] = \sum_{m=0}^{n} g[m]y[n-m]$

$= \sum_{m=0}^{n} g[m] \sum_{k=0}^{n-m} x[k]h[n-m-k]$. All sums have only a finite number of terms. Hence, the interchange of the order of the summations is justified and will give correct results.

B) Assumptions: $h[n]$ and $g[n]$ are stable sequences, and $x[n]$ is a bounded sequence with $|x[n]| \leq B < \infty$. Here, $y[m] = \sum_{k=-\infty}^{\infty} h[k]x[m-k] = \left(\sum_{k=k_1}^{k_2} h[k]x[m-k] \right)$ $+\varepsilon_{k_1,k_2}[m]$ with $|\varepsilon_{k_1,k_2}[m]| \leq \varepsilon_n B$. In this case, all sums have effectively only a finite number of terms and the error $|\varepsilon_{k_1,k_2}[m]|$ can be reduced by choosing k_1 and k_2 sufficiently large. As a result, in this case the problem is again effectively reduced to that of the one-sided sequences. Thus, the interchange of the order of the summations is again justified and will give correct results.

Hence, for the convolution to be associative, it is sufficient that the sequences be stable and single-sided.

2.57 $y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$. Since $h[k]$ is of length M and defined for $0 \leq k \leq M-1$, the convolution sum reduces to $y[n] = \sum_{k=0}^{(M-1)} x[n-k]h[k]$. $y[n]$ will be nonzero for all those values of n and for k which $n-k$ satisfies $0 \leq n-k \leq N-1$. Minimum value of $n-k=0$ and occurs for lowest n at $n=0$ and $k=0$. Maximum value of $n-k=N-1$ and occurs for maximum value of k at $M-1$. Thus $n-k=M-1$ $\Rightarrow n = N+M-2$. Hence the total number of nonzero samples $= N+M-1$.

2.58 $y[n] = \sum_{k=0}^{N-1} x[n-k]x[k]$. The maximum value of $y[n]$ occurs at $n=N-1$ when all product terms are present. The maximum value is given by
$y[N-1] = \sum_{k=0}^{N-1} a_{N-1-k} a_k$.

2.59 $y[n] = \sum_{k=0}^{N-1} x[n-k]h[k]$. The maximum value of $y[n]$ occurs at $n=N-1$ when all product terms are present. The maximum value is given by
$y[N-1] = \sum_{k=0}^{N-1} a_{N-1-k} b_k$.

2.60 **(a)** $y[n] = g_{ev}[n] \circledast h_{ev}[n] = \sum_{k=-\infty}^{\infty} h_{ev}[n-k] g_{ev}[k]$. Now,

$y[-n] = \sum_{k=-\infty}^{\infty} h_{ev}[-n-k] g_{ev}[k]$. Replace k by $-k$. Then the summation on the left becomes $y[-n] = \sum_{k=-\infty}^{\infty} h_{ev}[-n+k] g_{ev}[-k] = \sum_{k=-\infty}^{\infty} h_{ev}[-(n-k)] g_{ev}[-k]$
$= y[n]$. Hence $g_{ev}[n] \circledast h_{ev}[n]$ is an even sequence.

(b) $y[n] = g_{ev}[n] \circledast h_{od}[n] = \sum_{k=-\infty}^{\infty} h_{od}[n-k] g_{ev}[k]$. Now,

$y[-n] = \sum_{k=-\infty}^{\infty} h_{od}[-n-k] g_{ev}[k] = \sum_{k=-\infty}^{\infty} h_{od}[-n+k] g_{ev}[-k]$
$= \sum_{k=-\infty}^{\infty} h_{od}[-(n-k)] g_{ev}[-k] = -\sum_{k=-\infty}^{\infty} h_{od}[n-k] g_{ev}[k] = -y[n]$.
Hence $g_{ev}[n] \circledast h_{od}[n]$ is an odd sequence.

(c) $y[n] = g_{od}[n] \circledast h_{od}[n] = \sum_{k=-\infty}^{\infty} h_{od}[n-k] g_{od}[k]$. Now,

$y[-n] = \sum_{k=-\infty}^{\infty} h_{od}[-n-k] g_{od}[k] = \sum_{k=-\infty}^{\infty} h_{od}[-n+k] g_{od}[-k]$
$= \sum_{k=-\infty}^{\infty} h_{od}[-(n-k)] g_{od}[-k] = \sum_{k=-\infty}^{\infty} h_{od}[n-k] g_{od}[k] = y[n]$.
Hence $g_{od}[n] \circledast h_{od}[n]$ is an even sequence.

2.61 The impulse response of the cascade is given by $h[n] = h_1[n] \circledast h_2[n]$ where

$h_1[n] = \alpha^n \mu[n]$ and $h_2[n] = \beta^n \mu[n]$. Hence, $h[n] = \left(\sum_{k=0}^{n} \alpha^k \beta^{n-k}\right) \mu[n]$.

2.62 Now $h[n] = \alpha^n \mu[n]$. Therefore $y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k] = \sum_{k=0}^{\infty} \alpha^k x[n-k]$
$= x[n] + \sum_{k=1}^{\infty} \alpha^k x[n-k] = x[n] + \alpha \sum_{k=0}^{\infty} \alpha^k x[n-1-k] = x[n] + \alpha\, y[n-1]$.
Hence, $x[n] = y[n] - \alpha y[n-1]$. Thus the inverse system is given by
$y[n] = x[n] - \alpha x[n-1]$. The impulse response of the inverse system is given by
$h[n] = \{1,\ \alpha\},\ 0 \le n \le 1$.

2.63 From the results of Problem 2.62 we have $h[n] = \left(\sum_{k=0}^{n} \alpha^k \beta^{n-k}\right) \mu[n]$. Now,

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k] h[k] = \sum_{k=-\infty}^{\infty} \left(\sum_{m=0}^{k} \alpha^m \beta^{k-m}\right) \mu[m] x[n-k] = \sum_{k=0}^{\infty} \left(\sum_{m=0}^{k} \alpha^m \beta^{k-m}\right) x[n-k]$$

$$= x[n] + \sum_{k=1}^{\infty} \left(\sum_{m=0}^{k} \alpha^m \beta^{k-m}\right) x[n-k].$$ Substituting $r = k-1$ in the last expression we get

$$y[n] = x[n] + \sum_{r=0}^{\infty} \left(\sum_{m=0}^{r+1} \alpha^m \beta^{r+1-m}\right) x[n-r-1] = x[n] + \sum_{r=0}^{\infty} \left(\sum_{m=0}^{r} \alpha^m \beta^{r+1-m} + \alpha^{r+1}\right) x[n-r-1]$$

$$= x[n] + \beta \sum_{r=0}^{\infty} \left(\sum_{m=0}^{r} \alpha^m \beta^{r-m}\right) x[n-r-1] + \sum_{r=0}^{\infty} \alpha^{r+1} x[n-r-1]$$

$= x[n] + \beta y[n-1] + \alpha x[n-1] + \alpha^2 x[n-2] + \alpha^3 x[n-3] + \ldots$. The inverse system is therefore given by $x[n] = y[n] - (\alpha+\beta) y[n-1] + \alpha\beta y[n-2]$.

2.64 **(a)** $h[n] = h_1[n] \circledast h_2[n] \circledast h_3[n] \circledast h_3[n] + h_1[n] \circledast h_2[n] + h_3[n] \circledast h_4[n]$.

(b) $h[n] = h_4[n] + \dfrac{h_1[n] \circledast h_2[n] \circledast h_3[n]}{1 - h_1[n] \circledast h_2[n] \circledast h_5[n]}$.

2.65 $h[n] = h_1[n] \circledast h_2[n] + h_3[n]$. Now

$h_1[n] \circledast h_2[n] = (2\delta[n-2] - 3\delta[n+1]) \circledast (\delta[n-1] + 2\delta[n+2])$

$= 2\delta[n-2] \circledast \delta[n-1] - 3\delta[n+12] \circledast \delta[n-1] + 2\delta[n-2] \circledast \delta[n+2] - 3\delta[n+1] \circledast \delta[n+2]$

$= 2\delta[n-3] + \delta[n] - 6\delta[n-3]$. Therefore,
$h[n] = 2\delta[n-3] + \delta[n] - 6\delta[n-3] + 5\delta[n-5] + 7\delta[n-3] + 2\delta[n-1] - \delta[n] + 3\delta[n+1]$.

2.66 **(a)** The length of $x[n]$ is $8-4+1=5$. Using $x[n] = \dfrac{1}{h[0]}\left\{ y[n] - \sum_{k=0}^{3} h[k]x[n-k] \right\}$ we arrive at $\{x[n]\} = \{3,\ -2,\ 0,\ 1,\ 2\},\ 0 \le n \le 4$.

(b) The length of $x[n]$ is $7-4+1=4$. Using $x[n] = \dfrac{1}{h[0]}\left\{ y[n] - \sum_{k=0}^{3} h[k]x[n-k] \right\}$ we arrive at $\{x[n]\} = \{1,\ 2,\ 3,\ 4\},\ 0 \le n \le 3$.

(c) The length of $x[n]$ is $8-5+1=4$. Using $x[n] = \dfrac{1}{h[0]}\left\{ y[n] - \sum_{k=0}^{4} h[k]x[n-k] \right\}$ we arrive at $\{x[n]\} = \{1,\ -2,\ 3,\ -1\},\ 0 \le n \le 3$.

2.67 $y[n] = ay[n-1] + bx[n]$. Hence, $y[0] = ay[-1] + bx[0]$. Next, $y[1] = ay[0] + bx[1]$ $= a\big(ay[-1] + bx[0]\big) + bx[1] = a^2 y[-1] + abx[0] + bx[1]$. Continuing further in a similar way we obtained $y[n] = a^{n+1} y[-1] + \sum_{k=0}^{n} a^{n-k} bx[k]$.

(a) Let $y_1[n]$ be the output due to an input $x_1[n]$. Then
$y_1[n] = a^{n+1} y[-1] + \sum_{k=0}^{n} a^{n-k} bx_1[k]$. If $x_1[n] = x[n-n_o]$, then
$y_1[n] = a^{n+1} y[-1] + \sum_{k=0}^{n} a^{n-k} bx[k-n_o] = a^{n+1} y[-1] + \sum_{r=0}^{n-n_o} a^{n-n_o-r} bx[r]$.
However,
$y[n-n_o] = a^{n+1} y[-1] + \sum_{k=0}^{n} a^{n-k} bx[k-n_o] = a^{n-n_o+1} y[-1] + \sum_{r=0}^{n-n_o} a^{n-n_o-r} bx[r]$.
Hence $y_1[n] \ne y[n-n_o]$ if $y[-1] \ne 0$, i.e., the system is time-variant. The system is time-invariant if and only if $y[-1] = 0$, as then $y_1[n] = y[n-n_o]$.

(b) Let $y_1[n]$ and $y_1[n]$ be the outputs due to inputs $x_1[n]$ and $x_1[n]$, respectively. Let $y[n]$ be the output due to an input $\alpha x_1[n]+\beta x_2[n]$. However, $\alpha y_1[n]+\beta y_2[n]=$ $\alpha a^{n+1}y[-1]+\beta a^{n+1}y[-1]+\alpha\sum_{k=0}^{n}a^{n-k}bx_1[k]+\beta\sum_{k=0}^{n}a^{n-k}bx_2[k]$, whereas, $y[n]=a^{n+1}y[-1]+\alpha\sum_{k=0}^{n}a^{n-k}bx_1[k]+\beta\sum_{k=0}^{n}a^{n-k}bx_2[k]$. Hence, the system is nonlinear if $y[-1]\neq 0$ and is linear if and only if $y[-1]=0$.

(c) Generalizing the above result it can be shown that an N–th order causal discrete-time system is linear and time-invariant if and only if $y[-r]=0, 1\le r\le N$.

2.68 $y[n]=p_0x[n]+p_1x[n-1]-d_1y[n-1]$ leads to $x[n]=\frac{1}{p_0}y[n]+\frac{d_1}{p_0}y[n-1]-\frac{p_1}{p_0}x[n-1]$, which is the difference equation characterizing the inverse system.

2.69 $s[n]=\sum_{k=0}^{n}h[k]\mu[n-k]=\sum_{k=0}^{n}h[k],\ n\ge 0,$ and $s[n]=0, n<0$. Since $h[k]$ is nonnegative, $s[n]$ is a monotonically increasing function of n for $n\ge 0$, and is not oscillatory. Hence, there is no overshoot.

2.70 **(a)** $f[n]=f[n-1]+f[n-2]$. Let $f[n]=\alpha r^n$, then the difference equation reduces to $\alpha r^n-\alpha r^{n-1}-\alpha r^{n-2}=0$ which reduces further to $r^2-r-1=0$ resulting in $r=\frac{1\pm\sqrt{5}}{2}$. Thus, $f[n]=\alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^n+\alpha_1\left(\frac{1-\sqrt{5}}{2}\right)^n$.

As $f[0]=0,$ hence $\alpha_1+\alpha_2=0$. Also $f[1]=1,$, and hence

$\left(\frac{\alpha_1+\alpha_2}{2}\right)+\sqrt{5}\left(\frac{\alpha_1-\alpha_2}{2}\right)=1$. Solving for α_1 and α_2, we get $\alpha_1=-\alpha_2=\frac{1}{\sqrt{5}}$. Hence,

$$f[n]=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

(b) $y[n]=y[n-1]+y[n-2]+x[n-1]$. As the system is LTI, the initial conditions are equal to zero. Let $x[n]=\delta[n]$. Then $y[n]=y[n-1]+y[n-2]+\delta[n-1]$. Hence, $y[0]=y[-1]+y[-2]=0$ and $y[1]=y[0]+y[-2]+\delta[0]=1$. For $n>1$, the corresponding difference equation is $y[n]=y[n-1]+y[n-2]$ with initial conditions $y[0]=0$ and $y[1]=1$, which are the same as those for the solution of the Fibonacci's sequence. Hence $y[n]=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$. Thus denotes the impulse response of a causal LTI system described by the difference equation $y[n]=y[n-1]+y[n-2]+x[n-1]$.

2.71 $y[n]=\alpha y[n-1]+x[n]$. Denoting $y[n]=y_{re}[n]+j\,y_{im}[n]$, and $\alpha=a+jb$, we get

$y_{re}[n] + j\,y_{im}[n] = (a + j\,b)(y_{re}[n-1] + j\,y_{im}[n-1]) + x[n]$. Equating the real and the imaginary parts, and noting that $x[n]$ is real, we get $y_{re}[n] = ay_{re}[n-1] - by_{im}[n-1] + x[n]$, $y_{im}[n] = by_{re}[n-1] + ay_{im}[n-1]$. From the second equation we have $y_{im}[n-1] = \frac{1}{a} y_{im}[n] - \frac{b}{a} y_{re}[n-1]$. Substituting this equation in the top left equation we arrive at

$y_{re}[n] = ay_{re}[n-1] - \frac{b}{a} y_{im}[n] + \frac{b^2}{a} y_{re}[n-1] + x[n]$, from which we get

$by_{im}[n-1] = -ay_{re}[n-1] + (a^2 + b^2)y_{re}[n-2] + ax[n-1]$. Substituting this equation in the equation $y_{re}[n] = ay_{re}[n-1] - by_{im}[n-1] + x[n]$ we arrive at

$y_{re}[n] = 2ay_{re}[n-1] - (a^2 + b^2)y_{re}[n-2] + x[n] - ax[n-1]$ which is a second-order difference equation representing $y_{re}[n]$ in terms of $x[n]$.

2.72 The first-order causal LTI system is characterized by the difference equation $y[n] = p_0 x[n] + p_1 x[n-1] - d_1 y[n-1]$. Letting $x[n] = \delta[n]$ we obtain the difference equation representation of its impulse response $h[n] = p_0\delta[n] + p_1\delta[n-1] - d_1 h[n-1]$. Solving it for $n = 0,1,2,$ we get $h[0] = p_0, h[1] = p_1 - d_1 h[0] = p_1 - d_1 p_0$, and $h[2] = -d_1 h[1] = -d_1 p_0 (p_1 - d_1 p_0)$. Solving these equations we get $p_0 = h[0]$, $d_1 = -\frac{h[2]}{h[1]}$, and $p_1 = h[1] - \frac{h[2]h[0]}{h[1]}$.

2.73 $\sum_{k=0}^{M} p_k x[n-k] = \sum_{k=0}^{N} d_k y[n-k]$. Let $x[n] = \delta[n]$. Then $\sum_{k=0}^{M} p_k \delta[n-k] = \sum_{k=0}^{N} d_k h[n-k]$. Thus, $p_r = \sum_{k=0}^{N} d_k h[r-k]$. Since the system is assumed to be causal, $h[r-k] = 0$ for all $k > r$. Hence, $p_r = \sum_{k=0}^{N} d_k h[r-k] = \sum_{k=0}^{N} h[k] d_{r-k}$.

2.74 For a filter with a complex-valued impulse response, the first part of the proof is the same as that for a filter with a real-valued impulse response. From $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$ we get $|y[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right| \le \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|$. Since the input is bounded $|x[n]| \le B_x$. Therefore $|y[n]| \le B_x \sum_{k=-\infty}^{\infty} |h[k]|$. So if $\sum_{k=-\infty}^{\infty} |h[k]| = S < \infty$, then $|y[n]| \le B_x S$ indicating that $y[n]$ is also bounded.

To prove the converse we need to show that if a bounded input is produced by a bounded input then $S < \infty$. Consider the following bounded input defined by $x[n] = \frac{h^*[-n]}{|h[-n]|}$. Then $y[n] = \sum_{k=-\infty}^{\infty} \frac{h^*[-k]h[k]}{|h[k]|} = \sum_{k=-\infty}^{\infty} |h[k]| = S$. Now since the output is bounded, $S < \infty$. Thus for a filter with a complex impulse response is BIBO stable if and only if $\sum_{k=-\infty}^{\infty} |h[k]| = S < \infty$.

2.75 The impulse response of the cascade is $g[k] = \sum_{k=-\infty}^{\infty} h_1[k-r]h_2[r]$. Thus
$$\sum_{k=-\infty}^{\infty} |g[k]| = \sum_{k=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} |h_1[k-r]||h_2[r]| \le \left(\sum_{k=-\infty}^{\infty} |h_1[k]|\right)\left(\sum_{r=-\infty}^{\infty} |h_2[r]|\right).$$
Since $h_1[n]$ and $h_2[n]$ are stable, $\sum_k |h_1[k]| < \infty$ and $\sum_k |h_2[k]| < \infty$. Hence $\sum_k |g[k]| < \infty$ and as a result, a cascade of two stable LTI systems is also stable.

2.76 The impulse response of the parallel structure is $g[n] = h_1[n] + h_2[n]$. Now,
$$\sum_{k=-\infty}^{\infty} |g[k]| = \sum_{k=-\infty}^{\infty} |h_1[k] + h_2[k]| \le \sum_{k=-\infty}^{\infty} |h_1[k]| + \sum_{k=-\infty}^{\infty} |h_2[k]|.$$
Since $h_1[n]$ and $h_2[n]$ are stable, $\sum_k |h_1[k]| < \infty$ and $\sum_k |h_2[k]| < \infty$. Hence $\sum_k |g[k]| < \infty$ and as a result, a parallel connection of two stable LTI systems is also stable.

2.77 Consider a cascade connection of two passive LTI systems with an input $x[n]$ and an output $y[n]$. Let $y_1[n]$ and $y_2[n]$ be the outputs of the two systems for the input $x[n]$. Now $\sum_{n=-\infty}^{\infty} |y_1[n]|^2 \le \sum_{n=-\infty}^{\infty} |x[n]|^2$ and $\sum_{n=-\infty}^{\infty} |y_2[n]|^2 \le \sum_{n=-\infty}^{\infty} |x[n]|^2$. Let $y_1[n] = y_2[n] = x[n]$ satisfying the above inequalities. Then $y[n] = y_1[n] + y_2[n]$ $= 2x[n]$ and as a result, $\sum_{n=-\infty}^{\infty} |y[n]|^2 = 4\sum_{n=-\infty}^{\infty} |x[n]|^2 > \sum_{n=-\infty}^{\infty} |x[n]|^2$. Hence, the parallel connection of two passive LTI systems may not be passive.

2.78 Consider a parallel connection of two passive LTI systems with an input $x[n]$ and an output $y[n]$. Let $y_1[n]$ and $y_2[n]$ be the outputs of the two systems for the input $x[n]$. Now $\sum_{n=-\infty}^{\infty} |y_1[n]|^2 \le \sum_{n=-\infty}^{\infty} |x[n]|^2$ and $\sum_{n=-\infty}^{\infty} |y_2[n]|^2 \le \sum_{n=-\infty}^{\infty} |x[n]|^2$. Let $y_1[n] = y_2[n] = x[n]$ satisfying the above inequalities. Then $y[n] = y_1[n] + y_2[n]$ $= 2x[n]$ and as a result, $\sum_{n=-\infty}^{\infty} |y[n]|^2 = 4\sum_{n=-\infty}^{\infty} |x[n]|^2 > \sum_{n=-\infty}^{\infty} |x[n]|^2$. Hence, the parallel connection of two passive LTI systems may not be passive.

2.79 Let the difference equation $\sum_{k=0}^{M} p_k x[n-k] = y[n] + \sum_{k=1}^{N} d_k y[n-k]$ represents the causal IIR digital filter. For an input $x[n] = \delta[n]$, the corresponding output is then $y[n] = h[n]$, the impulse response of the filter. As the number of coefficients $\{p_k\}$ is $M+1$ and the number of coefficients $\{d_k\}$ is N, there are a total of $N+M+1$ unknowns. To determine these coefficients from the impulse response samples, we compute only the first $N+M+1$ impulse response samples. To illustrate the method, without any loss of generality, we assume $N = M = 3$. Then, from the difference equation we arrive at the following $N+M+1 = 7$ equations:

$h[0] = p_0,$

$h[1]+h[0]d_1 = p_1,$
$h[2]+h[1]d_1 + h[0]d_2 = p_2,$
$h[3]+h[2]d_1 + h[1]d_2 + h[0]d_2 = p_2,$
$h[4]+h[3]d_1 + h[2]d_2 + h[1]d_2 = 0,$
$h[5]+h[4]d_1 + h[3]d_2 + h[2]d_2 = 0,$
$h[6]+h[5]d_1 + h[4]d_2 + h[3]d_2 = 0.$

Writing the last three equations in matrix form we arrive at

$$\begin{bmatrix} h[4] \\ h[5] \\ h[6] \end{bmatrix} = \begin{bmatrix} h[3] & h[2] & h[1] \\ h[4] & h[3] & h[2] \\ h[5] & h[4] & h[3] \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ and hence, } \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = -\begin{bmatrix} h[3] & h[2] & h[1] \\ h[4] & h[3] & h[2] \\ h[5] & h[4] & h[3] \end{bmatrix}^{-1} \begin{bmatrix} h[4] \\ h[5] \\ h[6] \end{bmatrix}.$$

Substituting these values in the first four equations written in matrix form we get

$$\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} h[0] & 0 & 0 & 0 \\ h[1] & h[0] & 0 & 0 \\ h[2] & h[1] & h[0] & 0 \\ h[3] & h[2] & h[1] & h[0] \end{bmatrix} \begin{bmatrix} 1 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

2.80 $y[n] = y[-1] + \sum_{\ell=0}^{n} x[\ell] = y[-1] + \sum_{\ell=0}^{n} \ell\mu[\ell] = y[-1] + \sum_{\ell=0}^{n} \ell = y[-1] + \frac{n(n+1)}{2}.$

(a) For $y[-1]=0, y[n] = \frac{n(n+1)}{2}.$

(b) For $y[-1]=-2, y[n] = -2 + \frac{n(n+1)}{2} = \frac{n^2+n-4}{2}.$

2.81 $y(nT) = y((n-1)T) + \int_{(n-1)T}^{nT} x(\tau)d\tau = y((n-1)T) + T\cdot x((n-1)T).$ Therefore, the difference equation representation is given by $y[n] = y[n-1] + T\cdot x[n-1]$ where $y[n] = y(nT)$ and $x[n] = x(nT)$.

2.82 $y[n] = \frac{1}{n}\sum_{\ell=1}^{n} x[\ell] = \frac{1}{n}\sum_{\ell=1}^{n-1} x[\ell] + \frac{1}{n}x[n],\ n \ge 1.$ Now $y[n-1] = \frac{1}{n-1}\sum_{\ell=1}^{n-1} x[\ell], n \ge 1,$ i.e, $\sum_{\ell=1}^{n-1} x[\ell] = (n-1)y[n-1].$ Thus, the difference equation representation is given by $y[n] = \left(\frac{n-1}{n}\right)y[n-1] + \frac{1}{n}x[n].$

2.83 $y[n] - 0.35y[n-1] = 2.4\mu[n]$ with $y[-1]=3.$ The total solution is given by $y[n] = y_c[n] + y_p[n],$ where $y_c[n]$ is the complementary solution and $y_p[n]$ is the particular solution.

$y_c[n]$ is obtained by solving $y_c[n] - 0.35y_c[n-1] = 0.$ To this end we set $y_c[n] = \lambda^n,$ which yields $\lambda^n - 0.35\lambda^{n-1} = 0$ resulting in the solution $\lambda = 0.35.$ Hence $y_c[n] = \alpha(0.35)^n.$

For the particular solution we choose $y_p[n] = \beta$. Substituting this solution in the difference equation representing the system we get $\beta - 0.35\beta = 2.4\mu[n]$. For $n = 0$ we get $\beta - 0.35\beta = 2.4$, i.e., $(1-0.35)\beta = 2.4$ and hence $\beta = 2.4/0.65 = 48/13$.

Therefore $y[n] = y_c[n] + y_p[n] = \alpha(0.35)^n + \frac{48}{13}, n \geq 0$. For $n = -1$, we thus have $y[-1] = 3 = \alpha(0.35)^{-1} + \frac{48}{13}$ implying $\alpha = -0.2423$. The total solution is thus given by $y[n] = -0.2423(0.35)^n + \frac{48}{13}, n \geq 0$.

2.84 $y[n] - 0.3y[n-1] - 0.04y[n-2] = 3^n \mu[n]$ with $y[-1] = 2$ and $y[-2] = 1$. The total solution is given by $y[n] = y_c[n] + y_p[n]$, where $y_c[n]$ is the complementary solution and $y_p[n]$ is the particular solution.

$y_c[n]$ is obtained by solving $y_c[n] - 0.3y_c[n-1] - 0.04y_c[n-2] = 0$. To this end we set $y_c[n] = \lambda^n$, which yields $\lambda^n - 0.3\lambda^{n-1} - 0.04\lambda^{n-2} = 0$ resulting in the solutions $\lambda = 0.4$ or $\lambda = -0.1$. Hence $y_c[n] = \alpha_1(0.4)^n + \alpha_2(-0.1)^n$.

For the particular solution we choose $y_p[n] = \beta(3)^n$. Substituting this solution in the difference equation representing the system we get
$\beta(3)^n - 0.3\beta(3)^{n-1} - 0.04\beta(3)^{n-2} = 3^n \mu[n]$. For $n = 0$ we have
$\beta - 0.3\beta(3)^{-1} - 0.04\beta(3)^{-2} = 1$ which yields $\beta = 1.1166$. Therefore
$y[n] = y_c[n] + y_p[n] = \alpha_1(0.4)^n + \alpha_2(-0.1)^n + 1.1166(3)^n, n \geq 0$. For $n = -1$ and $n = -2$ we thus have $y[-1] = \alpha_1(0.4)^{-1} + \alpha_2(-0.1)^{-1} + 1.1166(3)^{-1} = 2$ and
$y[-2] = \alpha_1(0.4)^{-1}2 + \alpha_2(-0.1)^{-2} + 1.1166(3)^{-2} = 1$. Solving these two equations we get $\alpha_1 = 0.5489$ and $\alpha_2 = -0.0255$. Hence,
$y[n] = 0.5489(0.4)^n - 0.0255(-0.1)^n + 1.1166(3)^n, n \geq 0$.

2.85 $y[n] - 0.3y[n-1] - 0.04y[n-2] = x[n] + 2x[n-1]$ with $x[n] = 3^n \mu[n]$, $y[-1] = 2$ and $y[-2] = 1$. The total solution is given by $y[n] = y_c[n] + y_p[n]$, where $y_c[n]$ is the complementary solution and $y_p[n]$ is the particular solution. From the solution of Problem 2.84, the complementary solution is of the form $y_c[n] = \alpha_1(0.4)^n + \alpha_2(-0.1)^n$.

To determine $y_p[n]$ we observe that the it is given by the sum of the particular solution $y_{p1}[n]$ of the difference equation $y_1[n] - 0.3y_1[n-1] - 0.04y_1[n-2] = x[n] = 3^n \mu[n]$ and the particular solution $y_{p2}[n]$ of the difference equation

$y_2[n]-0.3y_2[n-1]-0.04y_2[n-2]=2x[n-1]=2\cdot 3^{n-1}\mu[n-1]$. From the solution of Problem 2.84, we have $y_{p1}[n]=\beta(3)^n$. Hence, $y_{p2}[n]=2\,y_{p1}[n-1]=2\beta(3)^{n-1}$.

Therefore, $y_p[n]=y_{p1}[n]+2y_{p2}[n]=\beta(3)^n+2\beta(3)^{n-1}=3^n\mu[n]+2\cdot 3^{n-1}\mu[n-1]$. For $n=1$ the above equation reduces to $3\beta+2\beta=3+6$. Thus, $\beta=9/5$. Therefore, the total solution is given by

$y[n]=y_c[n]+y_p[n]=\alpha_1(0.4)^n+\alpha_2(-0.1)^n+\frac{9}{5}(3)^n+\frac{18}{5}(3)^{n-1}$, $n\geq 0$. For $n=-1$ and $n=-2$ we thus have $y[-1]=\alpha_1(0.4)^{-1}+\alpha_2(-0.1)^{-1}+\frac{9}{5}(3)^{-1}+\frac{18}{5}(3)^{-2}=2$ and $y[-2]=\alpha_1(0.4)^{-2}+\alpha_2(-0.1)^{-2}+\frac{9}{5}(3)^{-2}+\frac{18}{5}(3)^{-3}=1$. Solving these two equations we get $\alpha_1=0.3413$ and $\alpha_2=-0.0147$. Hence,

$y[n]=0.3413(0.4)^n-0.0147(-0.1)^n+\frac{9}{5}(3)^n+\frac{18}{5}(3)^{n-1}$, $n\geq 0$.

2.86 $h[n]-0.35h[n-1]=\delta[n]$. The solution is given by $h[n]=h_c[n]+h_p[n]$, where $h_c[n]$ is the complementary solution and $h_p[n]$ is the particular solution. If $h[n]$ is the impulse response, then $h_p[n]=0$. From Problem 2.83 we note that $h_c[n]=\alpha(0.35)^n$. Thus, $h[0]-0.35h[-1]=h[0]=1$. This implies $\alpha=1$. Hence, $h[n]=(0.35)^n$, $n\geq 0$.

2.87 The overall system can be regarded as the cascade of two causal LTI systems:
S1: $y[n]-0.3y[n-1]-0.04y[n-2]=x_1[n]$ and **S**2: $x_1[n]=x[n]+2x[n-1]$.

The impulse response $h_1[n]$ of the system S1 can be found by solving the complementary solution of $h_1[n]-0.3h_1[n-1]-0.04h_1[n-2]=\delta[n]$. Let the complementary solution be $h_{1c}[n]=\lambda^n$, we have $\lambda^n-0.3\lambda^{n-1}-0.04\lambda^{n-2}=0$ hence $\lambda=\{0.4,\,-0.1\}$. Therefore, the impulse response $h_1[n]$ is given by $h_1[n]=h_{1c}[n]=A(0.4)^n+B(-0.1)^n, n\geq 0$. Solving constants $A,B,$ we get $A=0.8$ and $B=0.2$. Hence $h_1[n]=0.8(0.4)^n+0.2(-0.1)^n, n\geq 0.$.

The impulse response $h_2[n]$ of the system S2 is given by $h_2[n]=\delta[n]+2\delta[n-1]$..

The impulse response $h[n]$ of the overall system is

$h[n]=h_1[n]*h_2[n]=\left(0.8(0.4)^n+0.2(-0.1)^n\right)\mu[n]+2\left(0.8(0.4)^{n-1}+0.2(-0.1)^{n-1}\right)\mu[n-1]$
$=\delta[n]+1.92(0.4)^{n-1}\mu[n-1]+0.38(-0.1)^{n-1}\mu[n-1]$.

2.88 $h[n]=(-\alpha)^n\mu[n]$, $0<\alpha<1$. Step response is then given by $s[n]=h[n]\circledast\mu[n]$

$= (-\alpha)^n \mu[n] \circledast \mu[n] = \sum_{k=-\infty}^{\infty} (-\alpha)^k \mu[k]\mu[n-k] = \begin{cases} \sum_{k=0}^{n} (-\alpha)^k, & n \geq 0, \\ 0, & n < 0 \end{cases}$

$= \begin{cases} \frac{1-(-\alpha)^{n+1}}{1+\alpha}, & n \geq 0, \\ 0, & n < 0. \end{cases}$

2.89 Let $A_n = \left|n^K (\lambda_i)^n\right|$. Then $\frac{A_{n+1}}{A_n} = \left|\frac{n+1}{n}\right|^K \lambda_i$. Now $\lim_{n\to\infty} \left|\frac{n+1}{n}\right|^K = 1$. Since there exists a positive integer N_o such that for all $n > N_o$, $0 < \frac{A_{n+1}}{A_n} < \frac{1+\lambda_i}{2} < 1$. Hence $\sum_{n=0}^{\infty} A_n$ converges.

2.90 $\{x[n]\} = \{-4,\ 5,\ 1,\ -2,\ -3,\ 0,\ 2\}, -3 \leq n \leq 3,$
$\{y[n]\} = \{6,\ -3,\ -1,\ 0,\ 8,\ 7,\ -2\}, -1 \leq n \leq 5,$
$\{w[n]\} = \{3,\ 2,\ 2,\ -1,\ 0,\ -2,\ 5\}, 2 \leq n \leq 8.$

(a) $r_{xx}[\ell] = \sum_{n=-3}^{3} x[n]x[n-\ell], -6 \leq \ell \leq 6.$
$\{r_{xx}[\ell]\} = \{-8,\ 10,\ 14,\ -11,\ -23,\ -11,\ 59,\ -11,\ -23,\ -11,\ 14,\ 10,\ -8\}, -6 \leq \ell \leq 6,$
$r_{yy}[\ell] = \sum_{n=-5}^{5} x[n]x[n-\ell], -6 \leq \ell \leq 6.$
$\{r_{yy}[\ell]\} = \{-12,\ 48,\ 29,\ -31,\ -30,\ 27,\ 163,\ 27,\ -30,\ -31,\ 29,\ 48,\ -12\}, -6 \leq \ell \leq 6,$
$r_{ww}[\ell] = \sum_{n=-6}^{6} x[n]x[n-\ell], -6 \leq \ell \leq 6.$
$\{r_{ww}[\ell]\} = \{15,\ 4,\ 6,\ -12,\ 6,\ -2,\ 47,\ -2,\ 6,\ -12,\ 6,\ 4,\ 15\}, -6 \leq \ell \leq 6,$

(b) $r_{xy}[\ell] = \sum_{n=-5}^{3} x[n]y[n-\ell], -8 \leq \ell \leq 4.$
$\{r_{xy}[\ell]\} = \{8,\ -38,\ 1,\ 51,\ 4,\ -30,\ -68,\ 43,\ 31,\ -3,\ -20,\ -6,\ 12\}, -8 \leq \ell \leq 4,$
$r_{xw}[\ell] = \sum_{n=-8}^{3} x[n]y[n-\ell], -11 \leq \ell \leq 1.$
$\{r_{xw}[\ell]\} = \{-20,\ 33,\ -5,\ -8,\ -24,\ 7,\ 12,\ 12,\ -7,\ -14,\ -5,\ 4,\ 6\}, -11 \leq \ell \leq 1,$

2.91 **(a)** $x_1[n] = \alpha^n \mu[n]$. $r_{x_1x_1}[\ell] = \sum_{n=-\infty}^{\infty} x_1[n]x_1[n-\ell] = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] \alpha^{n-\ell} \mu[n-\ell]$

$= \sum_{n=0}^{\infty} \alpha^{2n-\ell} \mu[n-\ell] = \begin{cases} \sum_{n=0}^{\infty} \alpha^{2n-\ell}, & \ell < 0, \\ \sum_{n=\ell}^{\infty} \alpha^{2n-\ell}, & \ell \geq 0, \end{cases} = \begin{cases} \frac{\alpha^{-\ell}}{1-\alpha^2}, & \ell < 0, \\ \frac{\alpha^{\ell}}{1-\alpha^2}, & \ell \geq 0. \end{cases}$

Note for $\ell \geq 0, r_{x_1x_1}[\ell] = \frac{\alpha^\ell}{1-\alpha^2}$, and for $\ell < 0, r_{x_1x_1}[\ell] = \frac{\alpha^{-\ell}}{1-\alpha^2}$. Replacing ℓ with $-\ell$ in the second expression we get $r_{x_1x_1}[-\ell] = \frac{\alpha^{-(-\ell)}}{1-\alpha^2} = \frac{\alpha^\ell}{1-\alpha^2} = r_{x_1x_1}[\ell]$. Hence, $r_{x_1x_1}[\ell]$

is an even function of ℓ. Maximum value of $r_{x_1x_1}[\ell]$ occurs at $\ell = 0$ since α^{ℓ} is a decaying function for increasing when $|\alpha| < 1$.

(b) $x_2[n] = \begin{cases} 1, & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases}$ Now $r_{x_2x_2}[\ell] = \sum_{n=0}^{N-1} x_2[n-\ell]$, where

$x_2[n-\ell] = \begin{cases} 1, & \ell \le n \le N-1+\ell, \\ 0, & \text{otherwise.} \end{cases}$ Therefore, $r_{x_2x_2}[\ell] = \begin{cases} 0, & \text{for } \ell < -(N-1), \\ N+\ell, & \text{for } -(N-1) \le \ell \le 0, \\ N, & \text{for } \ell = 0, \\ N-\ell, & \text{for } 0 < N-\ell \le N-1, \\ 0, & \text{for } \ell > N-1. \end{cases}$

It follows from the above that $r_{x_2x_2}[\ell]$ is a triangular function of ℓ, and hence is an even function with a maximum value of N at $\ell = 0$.

2.92 **(a)** $x_1[n] = \cos\left(\frac{\pi n}{M}\right)$ where M is a positive integer. Period of $x_1[n]$ is $2M$, and hence $r_{x_1x_1}[\ell] = \frac{1}{2M}\sum_{n=0}^{2M-1} x_2[n]x_2[n+\ell] = \frac{1}{2M}\sum_{n=0}^{2M-1}\cos\left(\frac{\pi n}{M}\right)\cos\left(\frac{\pi(n+\ell)}{M}\right)$

$$= \frac{1}{2M}\sum_{n=0}^{2M-1}\cos\left(\frac{\pi n}{M}\right)\left\{\cos\left(\frac{\pi n}{M}\right)\cos\left(\frac{\pi \ell}{M}\right) - \sin\left(\frac{\pi n}{M}\right)\sin\left(\frac{\pi \ell}{M}\right)\right\}$$

$$= \frac{1}{2M}\cos\left(\frac{\pi \ell}{M}\right)\sum_{n=0}^{2M-1}\cos^2\left(\frac{\pi n}{M}\right).$$

Now

$$\sum_{n=0}^{2M-1}\cos^2\left(\frac{\pi n}{M}\right) = \sum_{n=0}^{N-1}\cos^2\left(\frac{4\pi n}{N}\right) = \frac{1}{2}\sum_{n=0}^{N-1}\left\{1 + \cos\left(\frac{4\pi n}{N}\right)\right\} = \frac{N}{2} + \frac{1}{2}\sum_{n=0}^{N-1}\cos\left(\frac{4\pi n}{N}\right).$$

Let $C = \sum_{n=0}^{N-1}\cos\left(\frac{4\pi n}{N}\right)$ and $S = \sum_{n=0}^{N-1}\sin\left(\frac{4\pi n}{N}\right)$. Then $C + jS = \sum_{n=0}^{N-1} e^{j(4\pi n/N)}$

$= \frac{e^{j4\pi} - 1}{e^{j4\pi/N} - 1} = 0.$ This implies $C = 0$. Thus $\sum_{n=0}^{2M-1}\cos^2\left(\frac{\pi n}{M}\right) = \frac{N}{2} = M.$

Hence, $r_{x_1x_1}[\ell] = \frac{M}{2M}\cos\left(\frac{\pi \ell}{M}\right) = \frac{1}{2}\cos\left(\frac{\pi \ell}{M}\right).$

(b) $\{x_2[n]\} = \langle n \rangle_6 = \{0,\ 1,\ 2,\ 3,\ 4,\ 5\},\ 0 \le n \le 5$. It is a periodic sequence with a period 6. Thus, $r_{x_2x_2}[\ell] = \frac{1}{6}\sum_{n=0}^{5} x_2[n]\langle x_2[n+\ell]\rangle_6,\ 0 \le \ell \le 5$. $r_{x_2x_2}[\ell]$ is also a periodic sequence with a period 6.

$r_{x_2x_2}[0] = \frac{1}{6}\left(x_2[0]x_2[0] + x_2[1]x_2[1] + x_2[2]x_2[2] + x_2[3]x_2[3] + x_2[4]x_2[4] + x_2[5]x_2[5]\right) = \frac{55}{6},$

$r_{x_2x_2}[1] = \frac{1}{6}\left(x_2[0]x_2[1] + x_2[1]x_2[2] + x_2[2]x_2[3] + x_2[3]x_2[4] + x_2[4]x_2[5] + x_2[5]x_2[0]\right) = \frac{40}{6},$

$$r_{x_2x_2}[2] = \frac{1}{6}\left(x_2[0]x_2[2] + x_2[1]x_2[3] + x_2[2]x_2[4] + x_2[3]x_2[5] + x_2[4]x_2[0] + x_2[5]x_2[1]\right) = \frac{32}{6},$$
$$r_{x_2x_2}[3] = \frac{1}{6}\left(x_2[0]x_2[3] + x_2[1]x_2[4] + x_2[2]x_2[5] + x_2[3]x_2[0] + x_2[4]x_2[1] + x_2[5]x_2[2]\right) = \frac{28}{6},$$
$$r_{x_2x_2}[4] = \frac{1}{6}\left(x_2[0]x_2[4] + x_2[1]x_2[5] + x_2[2]x_2[0] + x_2[3]x_2[1] + x_2[4]x_2[2] + x_2[3]x_2[4]\right) = \frac{31}{6},$$
$$r_{x_2x_2}[5] = \frac{1}{6}\left(x_2[0]x_2[5] + x_2[1]x_2[0] + x_2[2]x_2[1] + x_2[3]x_2[2] + x_2[4]x_2[3] + x_2[5]x_2[4]\right) = \frac{40}{6}.$$

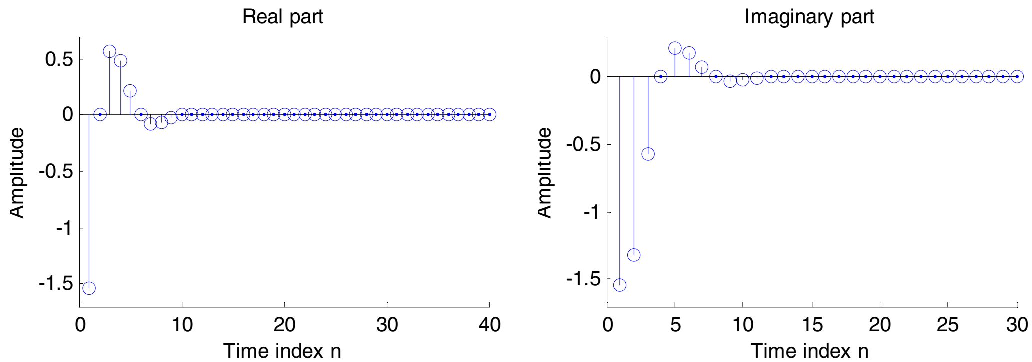
(c) $x_3[n] = (-1)^n$ is a periodic sequence with a period 2. Thus,
$r_{x_3x_3}[\ell] = \frac{1}{2}\sum_{n=0}^{1} x_3[n]x_3[n+\ell]$, $0 \le \ell \le 1$. Hence,
$r_{x_3x_3}[0] = \frac{1}{2}\left(x_3[0]x_3[0] + x_3[1]x_3[1]\right) = 1$, $r_{x_3x_3}[1] = \frac{1}{2}\left(x_3[0]x_3[1] + x_3[1]x_3[0]\right) = -1$.
$r_{x_3x_3}[\ell]$ is also a periodic sequence with a period 2.

M2.1 **(a)** The input data entered during the execution of Program 2_2.m are:

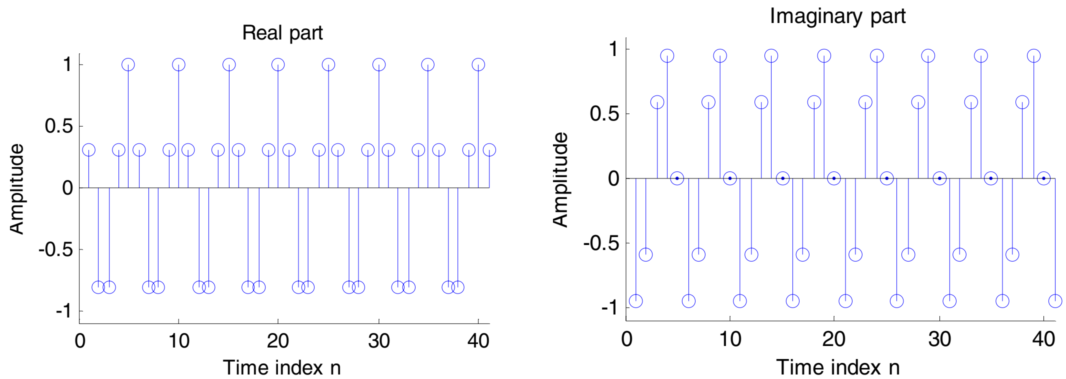
```
Type in real exponent = -1/12
Type in imaginary exponent = pi/6
Type in gain constant = 1
Type in length of sequence = 41
```

(b) The input data entered during the execution of Program 2_2.m are:

```
Type in real exponent = -1/12
Type in imaginary exponent = pi/6
Type in gain constant = 1
Type in length of sequence = 41
```



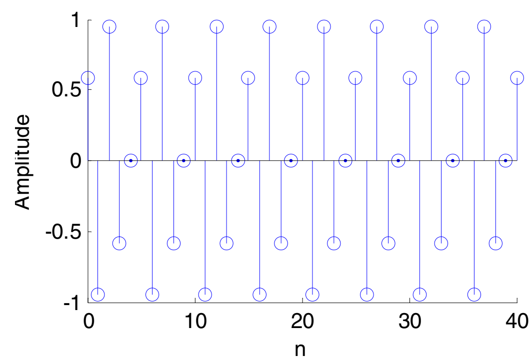
M2.2 **(a)** $\tilde{x}_a[n] = e^{-j0.4\pi n}$. The plots generated using Program 2_2.m are shown below:



(b) The code fragment used to generate $\tilde{x}_b[n] = \sin(0.8\pi n + 0.8\pi)$ is as follows:

```
x = sin(0.8*pi*n + 0.8*pi);
```

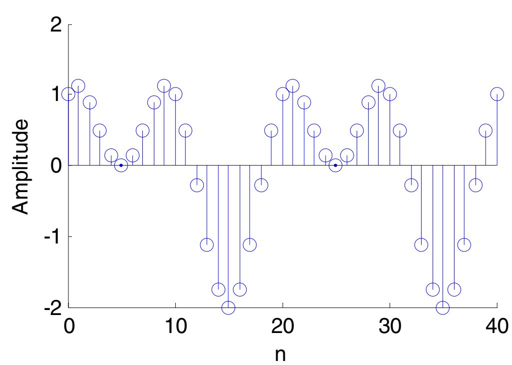
The plot of the periodic sequence is given below:



(c) The code fragment used to generate $\tilde{x}_c[n] = \mathrm{Re}\left(e^{j\pi n/5}\right) + \mathrm{Im}\left(e^{j\pi n/10}\right)$ is as follows:

```
x = real(exp(i*pi*n/5)+ imag(exp(i*pi*n/10);
```

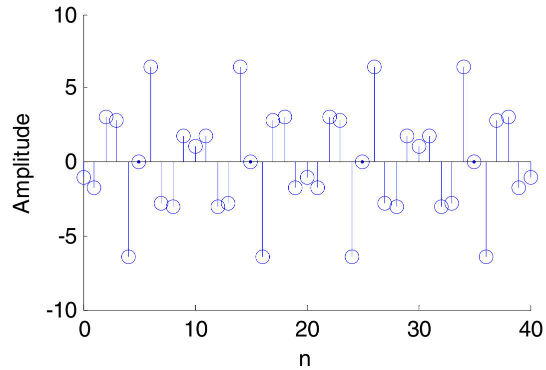
The plot of the periodic sequence is given below:



(d) The code fragment used to generate $\tilde{x}_d[n] = 3\cos(1.3\pi n) - 4\sin(0.5\pi n + 0.5\pi)$ is as follows:

```
x = 3*cos(1.3*pi*n)-4*sin(0.5*pi*n+0.5*pi);
```

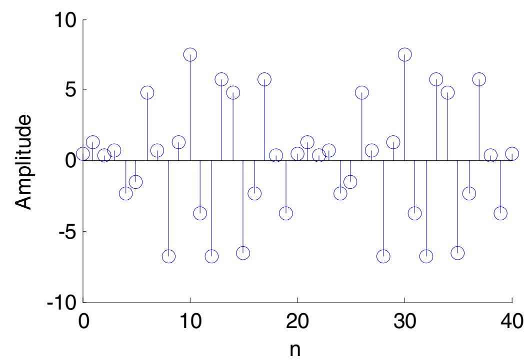
The plot of the periodic sequence is given below:



(e) The code fragment used to generate
$\tilde{x}_e[n] = 5\cos(1.5\pi n + 0.75\pi) + 4\cos(0.6\pi n) - \sin(0.5\pi n)$ is as follows:

```
x = 5*cos(1.5*pi*n+0.75*pi)+4*cos(0.6*pi*n)-sin(0.5*pi*n);
```

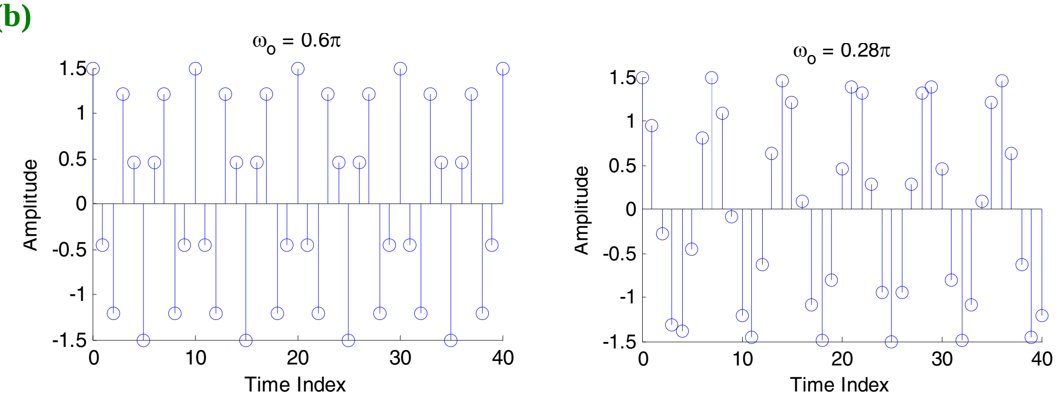
The plot of the periodic sequence is given below:

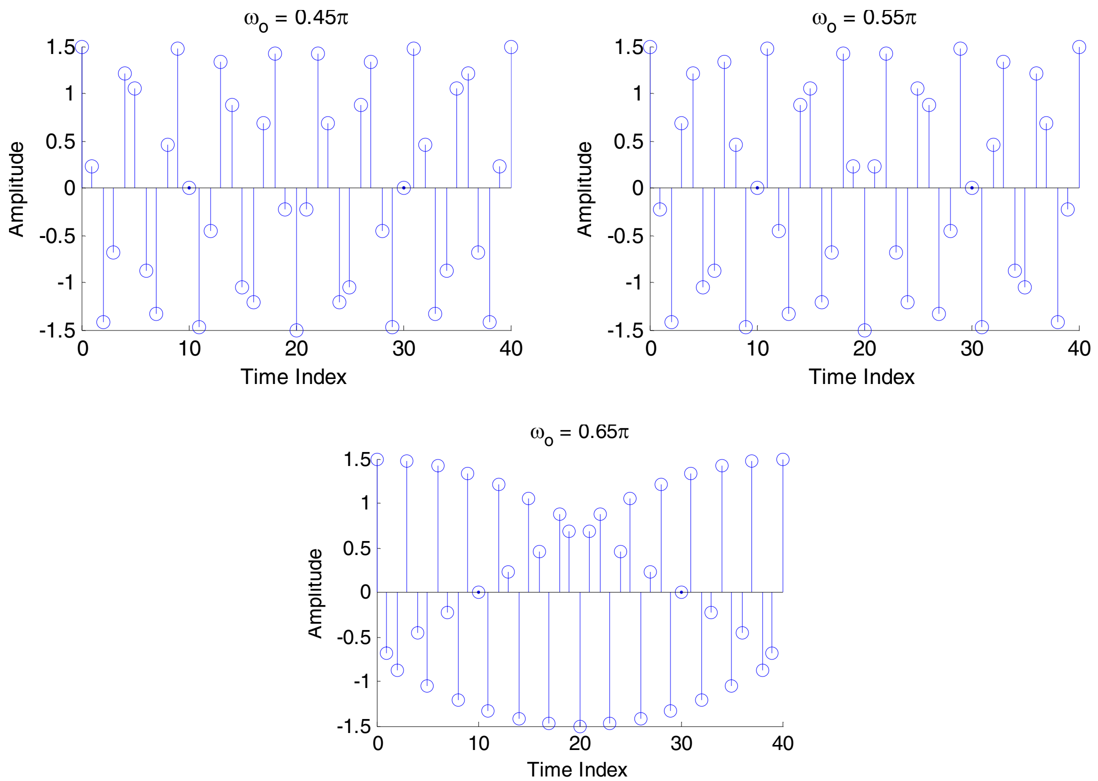


M2.3 **(a)**

```
L = input('Desired length = ');
A = input('Amplitude = ');
omega = input('Angular frequency = ');
phi = input('Phase = ');
n = 0:L-1;
x = A*cos(omega*n + phi);
stem(n,x);
xlabel('Time Index'); ylabel('Amplitude');
title(['\omega_{o} = ',num2str(omega/pi),'\pi']);
```

(b)





M2.4

```
t = 0:0.001:1;
fo = input('Frequency of sinusoid in Hz = ');
FT = input('Sampling frequency in Hz = ');
g1 = cos(2*pi*fo*t);
plot(t,g1,'-');
xlabel('time'); ylabel('Amplitude'); hold
n = 0:1:FT;
gs = cos(2*pi*fo*n/FT);
plot(n/FT,gs,'o'); hold off
```

M2.5

```
t = 0:0.001:0.85;
g1 = cos(6*pi*t); g2 = cos(14*pi*t); g3 =
cos(26*pi*t);
plot(t/0.85,g1,'-', t/0.85, g2, '--', t/0.85, g3,':');
xlabel('time'); ylabel('Amplitude'); hold
n = 0:1:8; gs = cos(0.6*pi*n); plot(n/8.5,gs,'o');
hold off
```

M2.6 As the length of the moving average filter is increased, the output of the filter gets more smoother. However, the delay between the input and the output sequences also increases (This can be seen from the plots generated by Program 2_4.m for various values of the filter length.)

M2.7

```
alpha = input('Alpha = ');
y0 = 1; y1 = 0.5*(y0 + (alpha/y0));
while abs(y1-y0)>0.00001
y2 = 0.5*(y1+(alpha/y1));
```

```
y0 = y1; y1 = y2;
end
disp('Squre root of alpha is'); disp(y1);
```

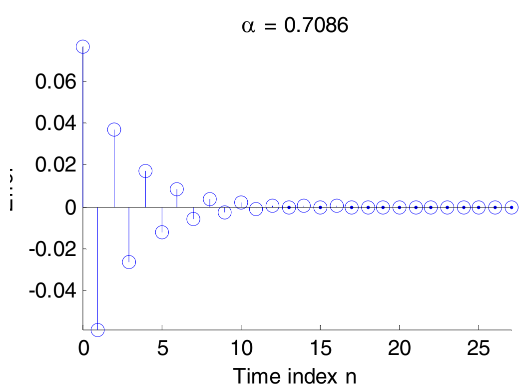
M2.8

```
format long
alpha = input('Alpha = ');
y0 = 0.3; y = zeros(1,61);
L = length(y) - 1;
y(1) = alpha - y0*y0 + y0; n = 2;

while abs(y(n-1) - y0) > 0.00001
     y2 = alpha - y(n-1)*y(n-1) + y(n-1);
     y0 = y(n-1); y(n) = y2;
     n = n+1;
end
disp('Square root of alpha is');disp(y(n-1));
m = 0:n-2;
err = y(1:n-1) - sqrt(alpha);
stem(m,err);
axis([0 n-2 min(err) max(err)]);
xlabel('Time index n'); ylabel('Error');
title(['\alpha = ',num2str(alpha)]);
```

The displayed out is

```
Square root of alpha is
0.84178104293115
```



M2.9 $\{x[n]\} = \{-4,\quad 5,\quad 1,\quad -2,\quad -3,\quad 0,\quad 2\}, -3 \le n \le 3,$

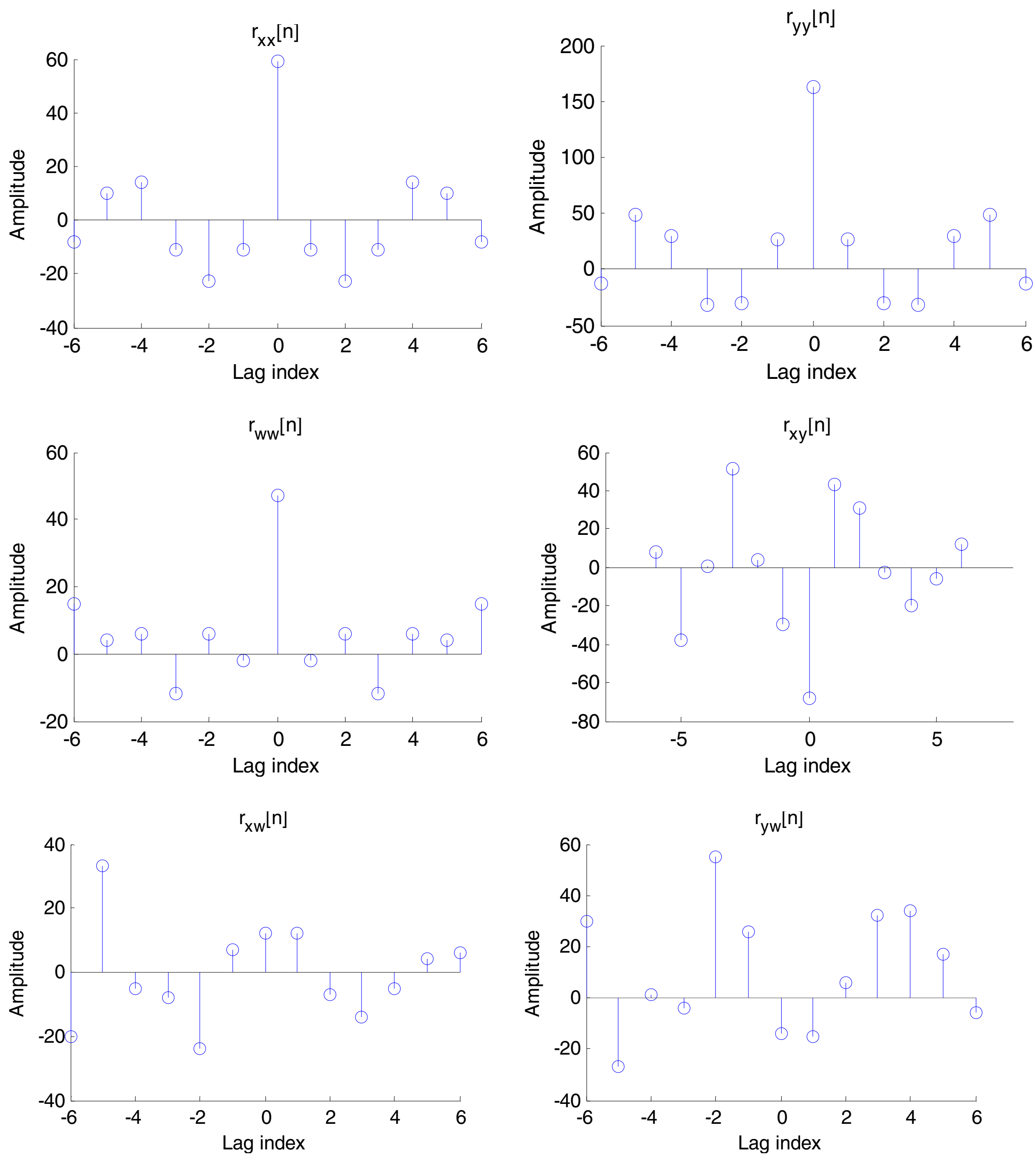
$\{y[n]\} = \{6,\quad -3,\quad -1,\quad 0,\quad 8,\quad 7,\quad -2\}, -1 \le n \le 5,$

$\{w[n]\} = \{3,\quad 2,\quad 2,\quad -1,\quad 0,\quad -2,\quad 5\}, 2 \le n \le 8.$

$\{r_{xx}[n]\} = \{-8,\quad 10,\quad 14,\quad -11,\quad -23,\quad -11,\quad 59,\quad -11,\quad -23,\quad -11,\quad 14,\quad 10,\quad -8\}, -6 \le n \le 6.$

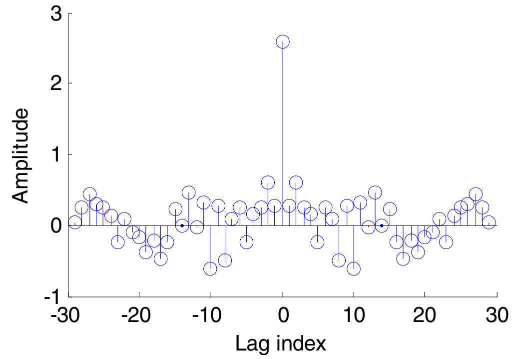
$\{r_{yy}[n]\} = \{-12,\quad 48,\quad 29,\quad -31,\quad -30,\quad -27,\quad 163,\quad -27,\quad -30,\quad -31,\quad 29,\quad 48,\quad -12\}, -6 \le n \le 6.$

$\{r_{ww}[n]\} = \{15,\quad 4,\quad 6,\quad -12,\quad 6,\quad -2,\quad 47,\quad -2,\quad 6,\quad -12,\quad 6,\quad 10,\quad 15\}, -6 \le n \le 6.$



M2.10

```
N = input('Length of sequence = ');
n = 0:N-1;
x = exp(-0.8*n);
y = rand(1,N)-0.5+x;
n1 = length(x)-1;
r = conv(y,fliplr(y));
k = (-n1):n1;
stem(k,r);
xlabel('Lag_index'); ylabel('Amplitude');
```


Amplitude
Lag index
3
2
1
0
-1
-30
-20
-10
0
10
20
30