Appendix L: Derivation of Similarity Transformations

L.1 Introduction

In Section 5.7 in the text we saw that systems can be represented with different state variables even though the transfer function relating the output to the input remains the same. The various forms of the state equations were found by manipulating the transfer function, drawing a signal-flow graph, and then writing the state equations from the signal-flow diagram. These systems are called *similar* systems. Although their state-space representations are different, similar systems have the same transfer function and hence the same poles or eigenvalues.

The question now arises whether we can make transformations among similar systems from one set of state equations to another without using the transfer function and signal-flow graphs. In this Appendix we will derive this transformation.

L.2 Expressing Any Vector in Terms of Basis Vectors

Let us begin by reviewing the representation of vector quantities in space. In Chapter 3, we learned that the state variables form the axes of the state space. Using a second-order system as an example, Figure L.1 shows two sets of axes, x_1x_2 and z_1z_2.[1]

Thus a state vector, $\mathbf{x}$, in state space can be written either in terms of the state variables or axes, x_1 and x_2, or if we call it $\mathbf{z}$, the state variables or axes, z_1 and z_2. In other words, the same vector is expressed in terms of different state variables. From this discussion we begin to see that the transformation from one set of state equations to another may be simply the transformation from one set of axes to another set of axes. Let us look further into this possibility by first clarifying the ways in which vectors can be represented in space.

Unit vectors, $\mathbf{U}_{\mathbf{x}_1}$, and $\mathbf{U}_{\mathbf{x}_2}$, which are collinear with the axes x_1 and x_2, form linearly independent vectors called *basis* vectors for the space, x_1x_2. Any vector in the space can be written in two ways. First, it can be written as a linear combination of the basis vectors. This linear combination implies vector summation of the basis vectors to form that vector. Second, any vector can be written in terms of its components along the axes. Summarizing these two ways of writing a vector, we have

$$\mathbf{x} = x_1\mathbf{U}_{\mathbf{x}_1} + x_2\mathbf{U}_{\mathbf{x}_2} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{L.1}$$

[1] These axes are shown to be *orthogonal* (90° to each other) for clarity. In general, the axes need be only linearly independent and are not necessarily at 90°. Linear independence precludes collinear axes.

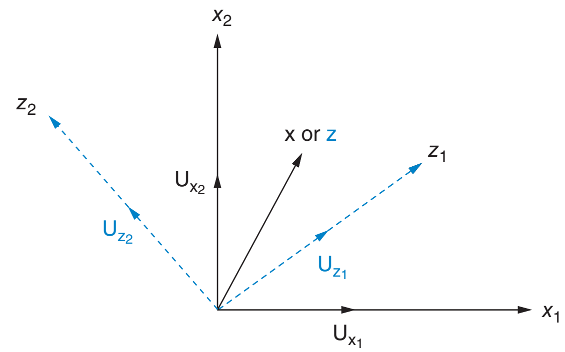

FIGURE L.1 State-space transformations

Similarly, the same vector, which will now be called **z**, can be written in terms of the basis vectors in the z_1z_2 space,

$$\mathbf{z} = z_1\mathbf{U}_{\mathbf{z}_1} + z_2\mathbf{U}_{\mathbf{z}_2} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{L.2}$$

L.3 Vector Transformations

What is the relationship between the components of **x** and **z** in Eqs. (L.1) and (L.2)? In other words, how do we transform vector **x** into vector **z** and vice versa? To begin we realize that unit vectors $\mathbf{U}_{\mathbf{z}_1}$, and $\mathbf{U}_{\mathbf{z}_2}$, which are collinear with z_1 and z_2 and are basis vectors for the space, z_1z_2, can be also written in terms of the basis vectors of the x_1x_2 space. Hence,

$$\mathbf{U}_{\mathbf{z}_1} = p_{11}\mathbf{U}_{\mathbf{x}_1} + p_{21}\mathbf{U}_{\mathbf{x}_2} \tag{L.3a}$$

$$\mathbf{U}_{\mathbf{z}_2} = p_{12}\mathbf{U}_{\mathbf{x}_1} + p_{22}\mathbf{U}_{\mathbf{x}_2} \tag{L.3b}$$

Substituting Eqs. (L.3) into Eq. (L.2), and realizing that the vectors **z** and **x** are the same, yields **x** in terms of the components of **z**, or

$$\mathbf{x} = (z_1p_{11} + z_2p_{12})\mathbf{U}_{\mathbf{x}_1} + (z_1p_{21} + z_2p_{22})\mathbf{U}_{\mathbf{x}_2} \tag{L.4}$$

which is equivalent to

$$\mathbf{x} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \mathbf{Pz} \tag{L.5}$$

and

$$\mathbf{z} = \mathbf{P}^{-1}\mathbf{x} \tag{L.6}$$

We can think of Eq. (L.5) as a transformation that takes **z** in the z_1z_2 plane and transforms it to **x** in the x_1x_2 plane. Hence, if we can find **P**, we can make the transformation between the two state-space representations.

L.4 Finding the Transformation Matrix, P

We can find the transformation matrix, **P**, from Eqs. (L.3). Since we know all vector quantities in the equation, we can then solve for p_{ij}'s. Notice that the columns of **P** are the coordinates of the basis vectors of the z_1z_2 space expressed as linear combinations of the basis vectors of the x_1x_2 space as shown in Eqs. (L.3). Thus the first column of **P** is $\mathbf{U}_{\mathbf{z}_1}$ and the second column is $\mathbf{U}_{\mathbf{z}_2}$. Partitioning **P**, we get

$$\mathbf{P} = \left[\mathbf{U}_{\mathbf{z}_1}\ \mathbf{U}_{\mathbf{z}_2}\right] \tag{L.7}$$

Let us look at an example of the transformation of a vector from one space to another.

Example L.1

Vector Transformations to New Basis

PROBLEM: Transform the vector

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \tag{L.8}$$

expressed with its basis vectors,

$$\mathbf{U}_{\mathbf{x}_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix};\quad \mathbf{U}_{\mathbf{x}_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix};\quad \mathbf{U}_{\mathbf{x}_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \tag{L.9}$$

to a vector expressed in the system,

$$\mathbf{U}_{\mathbf{z}_1} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix};\quad \mathbf{U}_{\mathbf{z}_2} = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix};\quad \mathbf{U}_{\mathbf{z}_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \tag{L.10}$$

SOLUTION: Using Eq. (L.2) as a guide, the vector **z** can be written in terms of the basis vectors, $\mathbf{U}_{\mathbf{z}_i}$.

$$\mathbf{z} = z_1\mathbf{U}_{\mathbf{z}_1} + z_2\mathbf{U}_{\mathbf{z}_2} + z_3\mathbf{U}_{\mathbf{z}_3} \tag{L.11}$$

Substituting the values of each $\mathbf{U}_{\mathbf{z}_i}$ given in Eq. (L.10) as components of the basis vectors, $\mathbf{U}_{\mathbf{x}_i}$, Eq. (L.11) is transformed to the components of **x**,

$$\mathbf{x} = z_1\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + z_2\begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + z_3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0z_1 + 0z_2 + 0z_3 \\ \left(1/\sqrt{2}\right)z_1 - \left(1/\sqrt{2}\right)z_2 + 0z_3 \\ \left(1/\sqrt{2}\right)z_1 + \left(1/\sqrt{2}\right)z_2 + 0z_3 \end{bmatrix} \tag{L.12}$$

which can be written as,

$$\mathbf{x} = \begin{bmatrix} 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \tag{L.13}$$

As we predicted, the columns of $\mathbf{P}$ are the basis vectors of the z_1z_2 space (Eq. (L.10)). Also,

$$\mathbf{z} = \mathbf{P}^{-1}\mathbf{x} = \begin{bmatrix} 0 & 0.707 & 0.707 \\ 0 & -0.707 & 0.707 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2.83 \\ 0 \\ 1 \end{bmatrix} \tag{L.14}$$

In summary, the vector $\mathbf{x} = [1 \quad 2 \quad 2]^T$ in the x_1x_2 space transforms into $\mathbf{z} = [2.83 \quad 0 \quad 1]^T$ in the z_1z_2 space. $\mathbf{x}$ and $\mathbf{z}$ are the same vector expressed in different coordinate systems.

Now that we are able to transform a state vector into different basis systems, let us see how to transform the state-space representation between basis systems.

L.5 Transforming the State Equations

We have seen that the same state vector can be expressed in terms of different basis vectors. This conversion amounts to selecting a different set of state variables to represent the same system transfer function.

Let us now convert a state-space representation with state vector, $\mathbf{x}$, into a state-space representation with a state vector, $\mathbf{z}$. Assume the state-space representation shown in Eq. (L.15).

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{L.15a}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{L.15b}$$

Let $\mathbf{x} = \mathbf{P}\mathbf{z}$ from Eq. (L.5). Hence,

$$\mathbf{P}\dot{\mathbf{z}} = \mathbf{A}\mathbf{P}\mathbf{z} + \mathbf{B}\mathbf{u} \tag{L.16a}$$

$$\mathbf{y} = \mathbf{C}\mathbf{P}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{L.16b}$$

Premultiplying the state equation by $\mathbf{P}^{-1}$,

$$\dot{\mathbf{z}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{z} + \mathbf{P}^{-1}\mathbf{B}\mathbf{u} \tag{L.17a}$$

$$\mathbf{y} = \mathbf{C}\mathbf{P}\mathbf{z} + \mathbf{D}\mathbf{u} \tag{L.17b}$$

Eqs. (L.17) are an alternate representation of a system in state space. The transformed system matrix is $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, the input coupling matrix is $\mathbf{P}^{-1}\mathbf{B}$, the output matrix is $\mathbf{C}\mathbf{P}$, and the feedforward matrix remains $\mathbf{D}$.

We now will show that the transfer function, $T(s) = Y(s)/U(s)$, which relates the output of the system to its input for the system represented by Eqs. (L.17), is the same as the system of Eqs. (L.15) if, $\mathbf{y}$ and $\mathbf{u}$ are scalars, $y(t)$ and $u(t)$.

From Eq. (3.73), the transfer function for the system of Eqs. (L.15) is

$$T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \tag{L.18}$$

The transfer function of the system of Eqs. (L.17) can be found by substituting its equivalent output, system, input, and feedforward matrices into Eq. (L.18). Hence, the transfer function for the system of Eqs. (L.17) is

$$T(s) = \frac{Y(s)}{U(s)} = \mathbf{CP}(s\mathbf{I} - \mathbf{P}^{-1}\mathbf{AP})^{-1}\mathbf{P}^{-1}\mathbf{B} + \mathbf{D} \tag{L.19}$$

Making successive use of the matrix inverse theorem, $(\mathbf{MN})^{-1} = \mathbf{N}^{-1}\mathbf{M}^{-1}$, we find

$$T(s) = \mathbf{CP}[\mathbf{P}(s\mathbf{I} - \mathbf{P}^{-1}\mathbf{AP})]^{-1}\mathbf{B} + \mathbf{D} = \mathbf{C}[\mathbf{P}(s\mathbf{I} - \mathbf{P}^{-1}\mathbf{AP})\mathbf{P}^{-1}]\mathbf{B} + \mathbf{D} \tag{L.20}$$

Since $(s\mathbf{I} - \mathbf{P}^{-1}\mathbf{AP})\mathbf{P}^{-1} = (s\mathbf{P}^{-1} - \mathbf{P}^{-1}\mathbf{AP})$,

$$T(s) = \mathbf{C}[\mathbf{P}(s\mathbf{P}^{-1} - \mathbf{P}^{-1}\mathbf{A})]^{-1}\mathbf{B} + \mathbf{D} = \mathbf{C}[(s\mathbf{I} - \mathbf{A})]^{-1}\mathbf{B} + \mathbf{D} \tag{L.21}$$

which is identical to Eq. (L.18). Since the transfer function is the same, the system's poles and zeros remain the same through the transformation.

We can show more formally that the eigenvalues do not change under a similarity transformation. The characteristic equation for the system prior to the transformation is $\det(s\mathbf{I}-\mathbf{A}) = 0$. After the transformation, the characteristic equation is $\det(s\mathbf{I}-\mathbf{P}^{-1}\mathbf{AP}) = 0$. But, $\mathbf{I} = \mathbf{P}^{-1}\mathbf{P}$. Therefore the characteristic equation after the transformation can be written as

$$\det(s\mathbf{P}^{-1}\mathbf{P} - \mathbf{P}^{-1}\mathbf{AP}) = \det[\mathbf{P}^{-1}(s\mathbf{I} - \mathbf{A})\mathbf{P}] = 0 \tag{L.22}$$

Since the determinant of the product of matrices is the product of the determinants,

$$\det[\mathbf{P}^{-1}(s\mathbf{I} - \mathbf{A})\mathbf{P}] = \det(\mathbf{P}^{-1})\det(s\mathbf{I} - \mathbf{A})\det(\mathbf{P}) = 0 \tag{L.23}$$

But,

$$\det(\mathbf{P}^{-1})\det(\mathbf{P}) = \det(\mathbf{I}) = 1 \tag{L.24}$$

Hence,

$$\det(s\mathbf{I} - \mathbf{P}^{-1}\mathbf{AP}) = \det(s\mathbf{I} - \mathbf{A}) = 0 \tag{L.25}$$

Eq. (L.25) shows that the eigenvalues do not change under the transformation.

In this appendix we have shown that a vector, $\mathbf{x}$, in the x_1x_2 basis system can be expressed as a vector, $\mathbf{z}$, in the z_1z_2 basis system using

$$\mathbf{x} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \mathbf{Pz} \tag{L.26}$$

Similarly, the inverse is

$$\mathbf{z} = \mathbf{P}^{-1}\mathbf{x} \tag{L.27}$$

We found that the transformation matrix, $\mathbf{P}$, consists of columns, which are the coordinates of the basis vectors of the z_1z_2 space expressed as linear combinations of the basis vectors of the x_1x_2 space, or

$$\mathbf{P} = \begin{bmatrix} \mathbf{U}_{\mathbf{z}_1} & \mathbf{U}_{\mathbf{z}_2} \end{bmatrix} \tag{L.28}$$

Using the previous results, the state equations can be transformed from the **x** state variables to the **z** state variables using

$$\dot{\mathbf{z}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{z} + \mathbf{P}^{-1}\mathbf{B}\mathbf{u} \tag{L.29a}$$

$$\mathbf{y} = \mathbf{C}\mathbf{P}\mathbf{z} + \mathbf{D}\mathbf{u} \tag{L.29b}$$

Finally, we found that the eigenvalues of the **x** system are the same as those of the **z** system. Hence, the transfer function calculated from either system will be the same.

Bibliography

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