Digital Communications I: Modulation and Coding Course

Spring - 2015

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Lecture 3b: Detection and Signal Spaces

Last time we talked about:

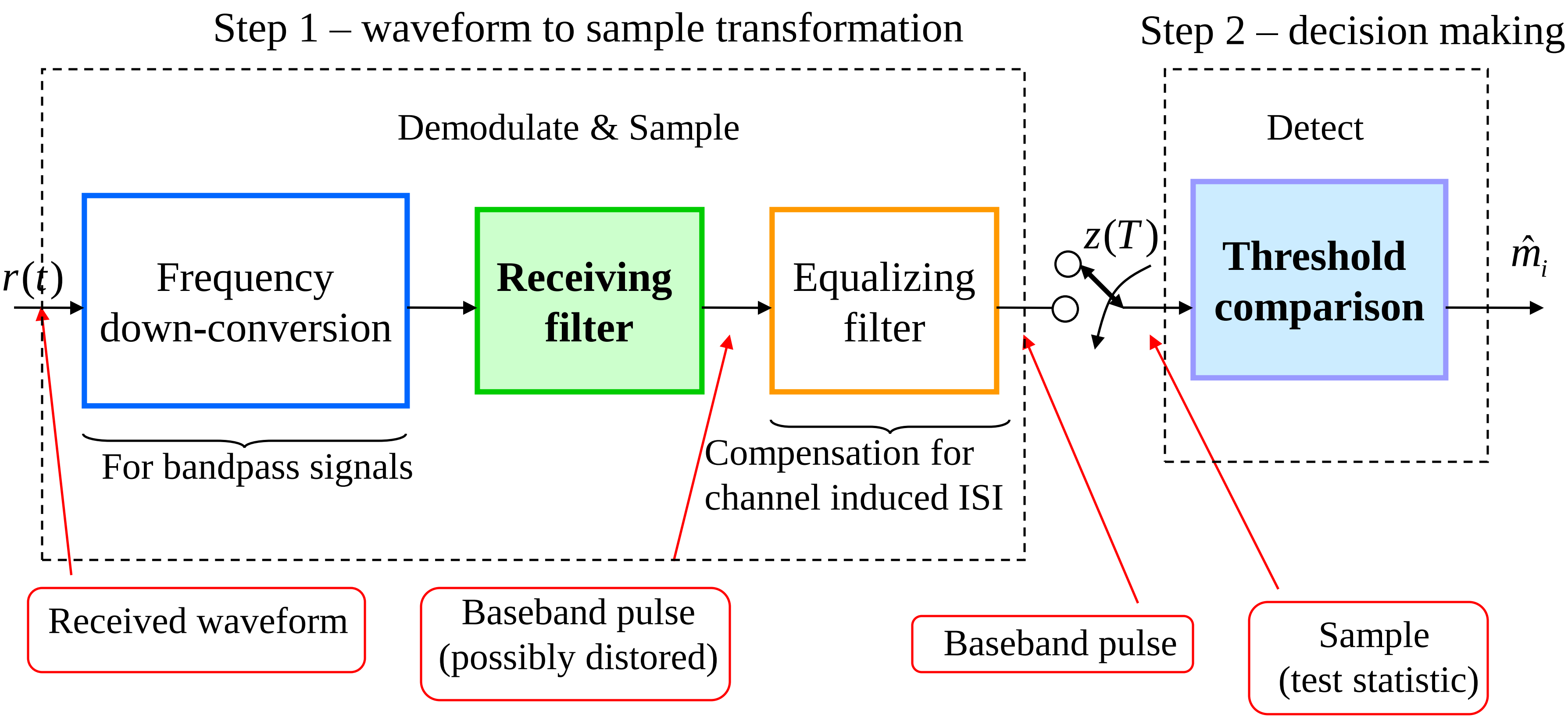
- Receiver structure
- Impact of AWGN and ISI on the transmitted signal
- Optimum filter to maximize SNR
  - Matched filter receiver and Correlator receiver

Receiver job

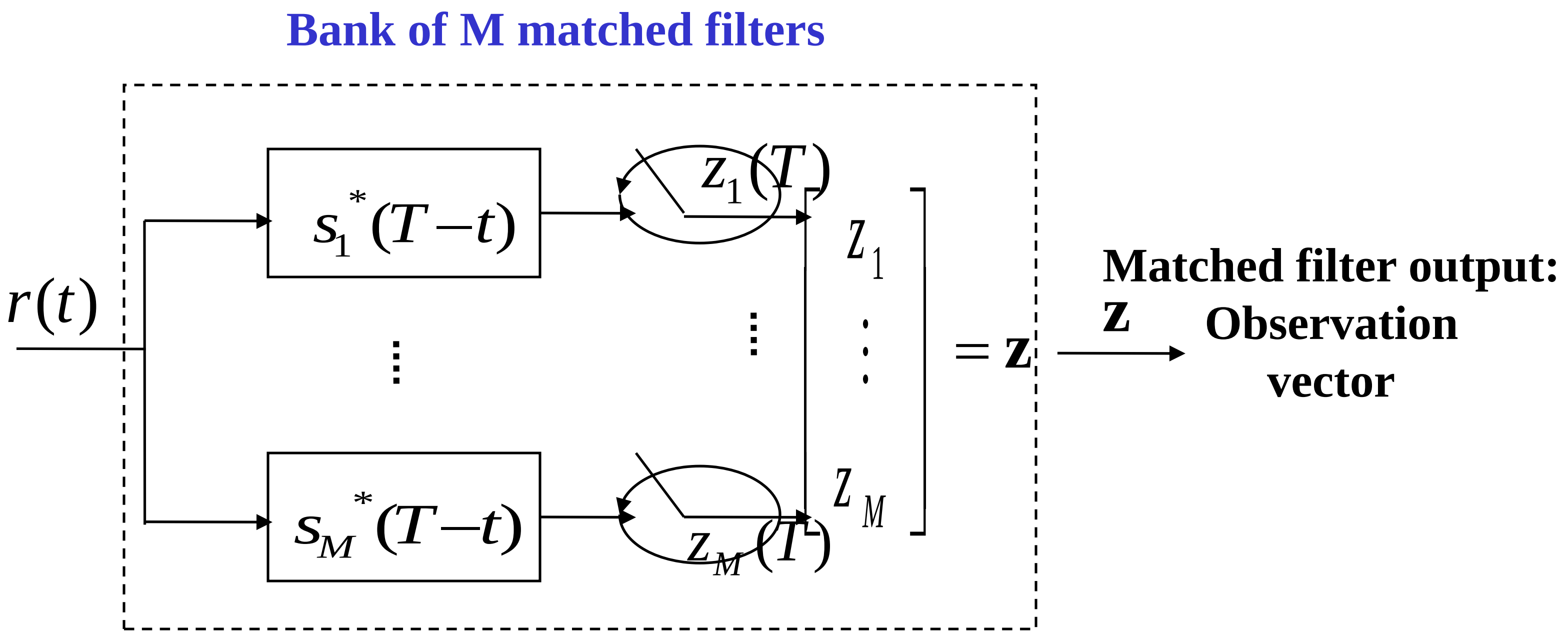
- Demodulation and sampling:
  - Waveform recovery and preparing the received signal for detection:
    - Improving the signal power to the noise power (SNR) using matched filter
    - Reducing ISI using equalizer
    - Sampling the recovered waveform
- Detection:
  - Estimate the transmitted symbol based on the received sample

Receiver structure

Digital Receiver



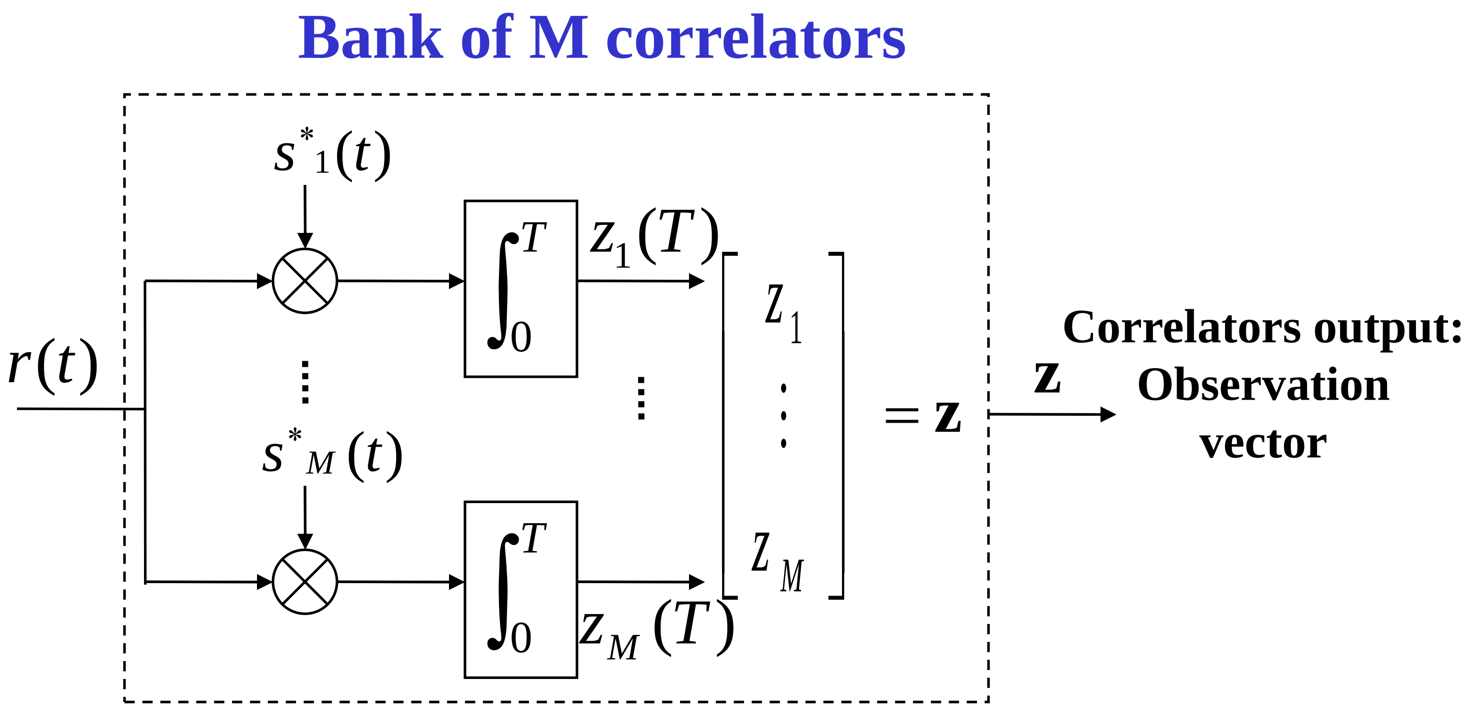
Implementation of matched filter receiver



$$z_i = r(t) * s^*_i(T-t) \quad i = 1,\dots,M$$

$$\mathbf{z} = (z_1(T), z_2(T), \dots, z_M(T)) = (z_1, z_2, \dots, z_M)$$

Implementation of correlator receiver



$$\mathbf{z} = (z_1(T), z_2(T), \ldots, z_M(T)) = (z_1, z_2, \ldots, z_M)$$

$$z_i = \int_0^T r(t) s_i(t) dt \quad i = 1, \ldots, M$$

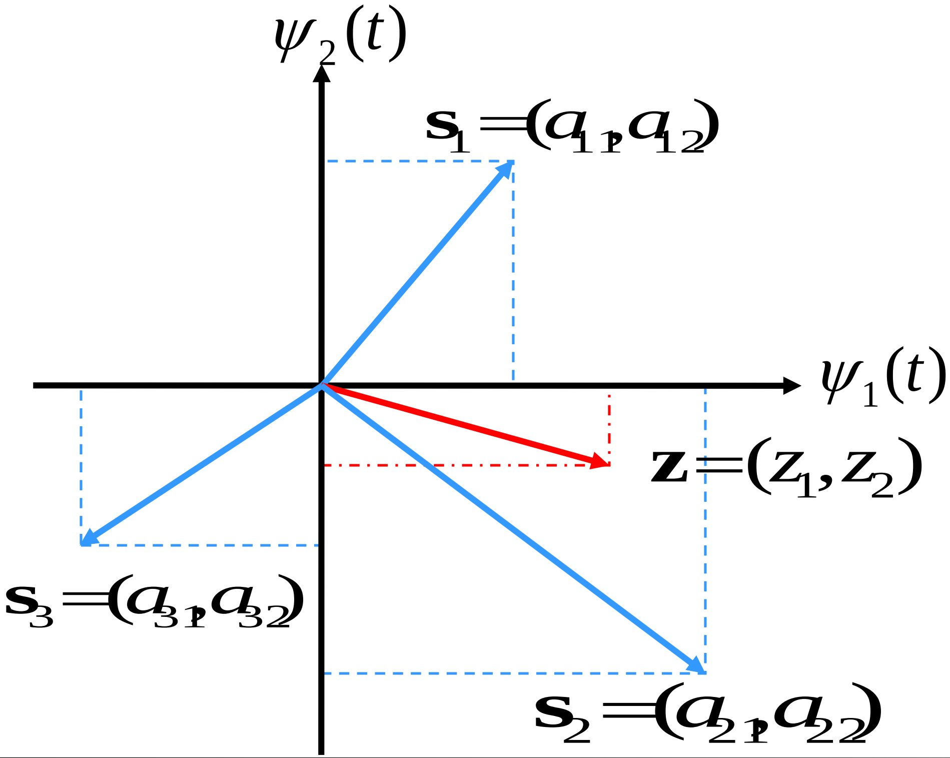
Today, we are going to talk about:

- Detection:
  - Estimate the transmitted symbol based on the received sample
- Signal space used for detection
  - Orthogonal N-dimensional space
  - Signal to waveform transformation and vice versa

Signal space

- What is a signal space?
  - Vector representations of signals in an N-dimensional orthogonal space
- Why do we need a signal space?
  - It is a means to convert signals to vectors and vice versa.
  - It is a means to calculate signals energy and Euclidean distances between signals.
- Why are we interested in Euclidean distances between signals?
  - For detection purposes: The received signal is transformed to a received vector. The signal which has the minimum Euclidean distance to the received signal is estimated as the transmitted signal.

Schematic example of a signal space



Transmitted signal alternatives

$$s_1(t) = a_{11}\psi_1(t) + a_{12}\psi_2(t) \Leftrightarrow \mathbf{s}_1 = (a_{11}, a_{12})$$

$$s_2(t) = a_{21}\psi_1(t) + a_{22}\psi_2(t) \Leftrightarrow \mathbf{s}_2 = (a_{21}, a_{22})$$

$$s_3(t) = a_{31}\psi_1(t) + a_{32}\psi_2(t) \Leftrightarrow \mathbf{s}_3 = (a_{31}, a_{32})$$

Received signal at matched filter output

$$z(t) = z_1\psi_1(t) + z_2\psi_2(t) \Leftrightarrow \mathbf{z} = (z_1, z_2)$$

Signal space

- To form a signal space, first we need to know the <u>inner product</u> between two signals (functions):
  - Inner (scalar) product:

$$< x(t), y(t) >= \int_{-\infty}^{\infty} x(t) y^*(t) dt$$

Analogous to the "dot" product of discrete n-space vectors
= cross-correlation between x(t) and y(t)

  - Properties of inner product:

$$< ax(t), y(t) >= a < x(t), y(t) >$$

$$< x(t), ay(t) >= a^* < x(t), y(t) >$$

$$< x(t) + y(t), z(t) >=< x(t), z(t) > + < y(t), z(t) >$$

Signal space …

- The distance in signal space is measure by calculating the norm.
- What is norm?
  - Norm of a signal:

$$\|x(t)\| = \sqrt{< x(t), x(t) >} = \sqrt{\int_{-\infty}^{\infty} |x(t)|^2 dt} = \sqrt{E_x}$$

= "length" or amplitude of x(t)

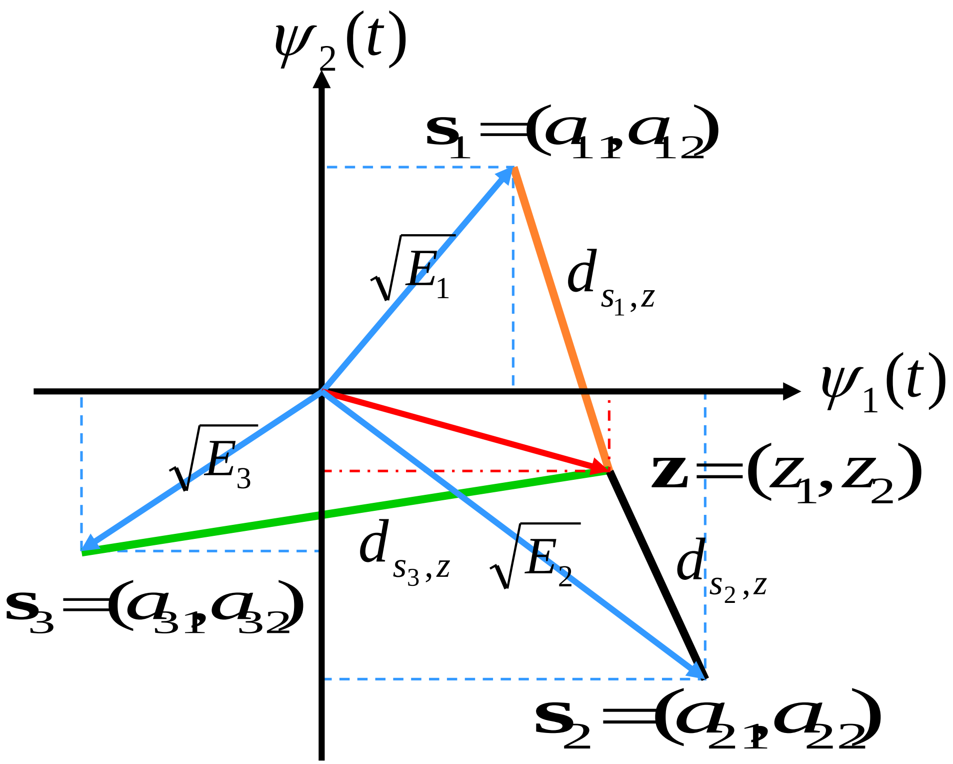
$$\|ax(t)\| = |a| \|x(t)\|$$

  - Norm between two signals:

$$d_{x,y} = \|x(t) - y(t)\|$$

- We refer to the norm between two signals as the Euclidean distance between two signals.

Example of distances in signal space



The Euclidean distance between signals *z(t)* and *s(t)*:

$$d_{s_i,z} = \| s_i(t) - z(t) \| = \sqrt{(a_{i1} - z_1)^2 + (a_{i2} - z_2)^2}$$
$$i = 1,2,3$$

Orthogonal signal space

- N-dimensional orthogonal signal space is characterized by N linearly independent functions $\{\psi_j(t)\}_{j=1}^{N}$ called basis functions. The basis functions must satisfy the orthogonality condition

$$< \psi_i(t), \psi_j(t) > = \int_0^T \psi_i(t)\psi_j^*(t)dt = K_i\delta_{ji} \qquad \begin{matrix} 0 \le t \le T \\ j, i = 1, \ldots, N \end{matrix}$$

where

$$\delta_{ij} = \begin{cases} 1 \rightarrow i = j \\ 0 \rightarrow i \neq j \end{cases}$$

- If all $K_i = 1$, the signal space is orthonormal.
- See my notes on Fourier Series

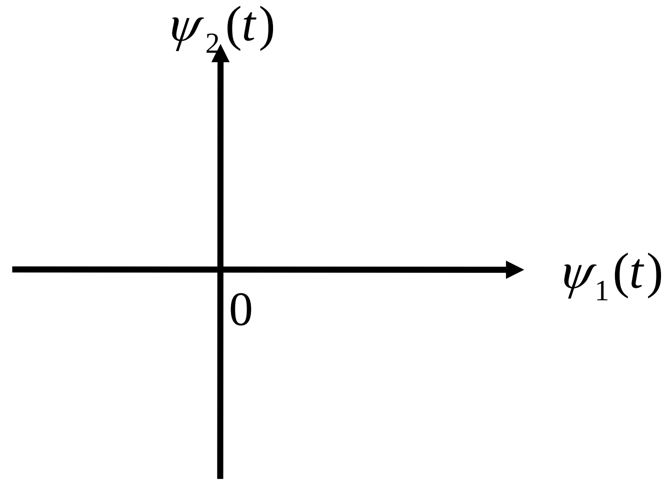
Example of an orthonormal basis

- Example: 2-dimensional orthonormal signal space

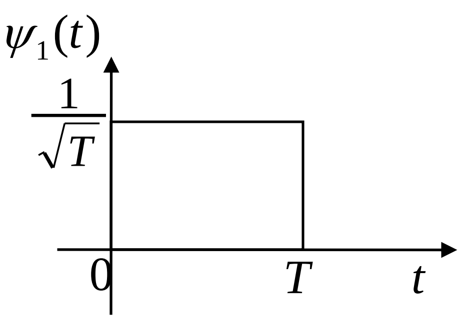
$$\begin{cases} \psi_1(t) = \sqrt{\dfrac{2}{T}}\cos(2\pi t/T) & 0 \le t < T \\ \psi_2(t) = \sqrt{\dfrac{2}{T}}\sin(2\pi t/T) & 0 \le t < T \end{cases}$$

$$<\psi_1(t), \psi_2(t)> = \int_0^T \psi_1(t)\psi_2(t)\,dt = 0$$

$$\|\psi_1(t)\| = \|\psi_2(t)\| = 1$$



- Example: 1-dimensional orthonormal signal space



$$\|\psi_1(t)\| = 1$$

Signal space ...

- Any arbitrary finite set of waveforms $\{s_i(t)\}_{i=1}^{M}$

  where each member of the set is of duration *T*, can be expressed as a linear combination of N orthonogal waveforms $\{\psi_j(t)\}_{j=1}^{N}$ where $N \le M$.

$$s_i(t) = \sum_{j=1}^{N} a_{ij}\psi_j(t) \qquad \begin{array}{l} i = 1,\ldots,M \\ N \le M \end{array}$$

where

$$a_{ij} = \frac{1}{K_j} < s_i(t), \psi_j(t) > = \frac{1}{K_j}\int_0^T s_i(t)\psi_j^*(t)dt \qquad \begin{array}{l} j = 1,\ldots,N \\ i = 1,\ldots,M \end{array} \quad 0 \le t \le T$$

$$\mathbf{s}_i = (a_{i1}, a_{i2}, \ldots, a_{iN})$$

Vector representation of waveform

$$E_i = \sum_{j=1}^{N} K_j \left|a_{ij}\right|^2$$

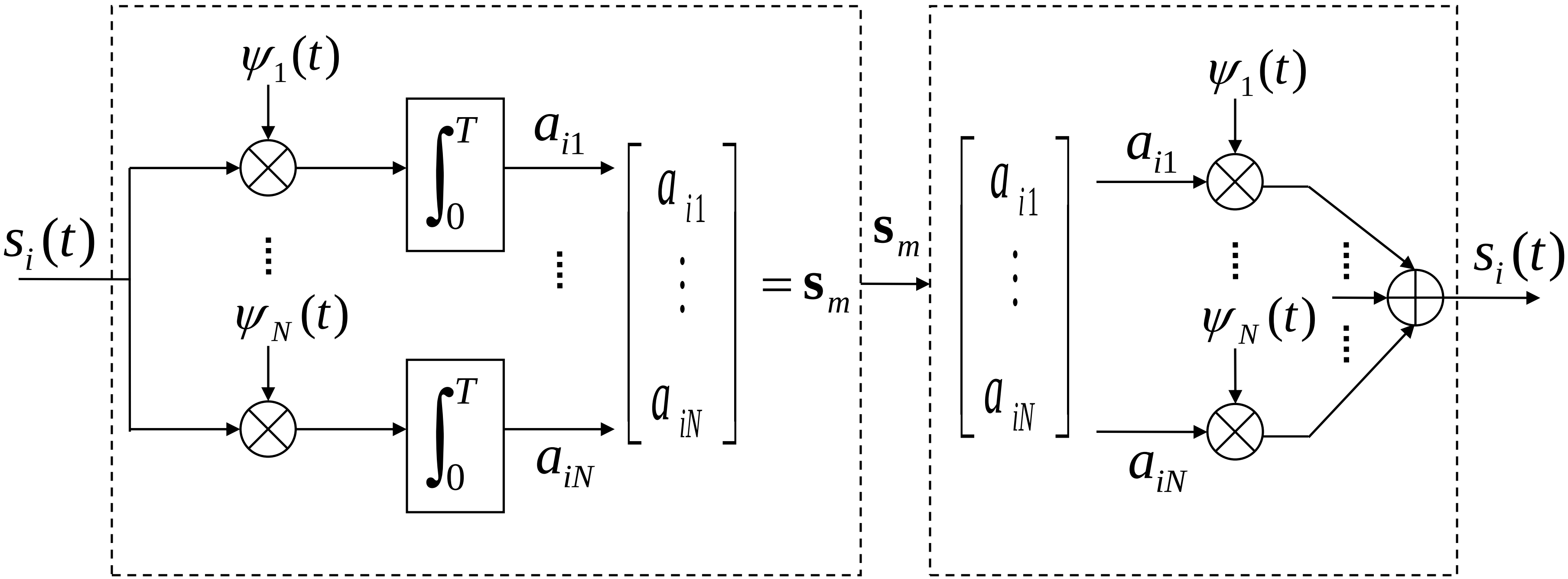
Waveform energy

Signal space ...

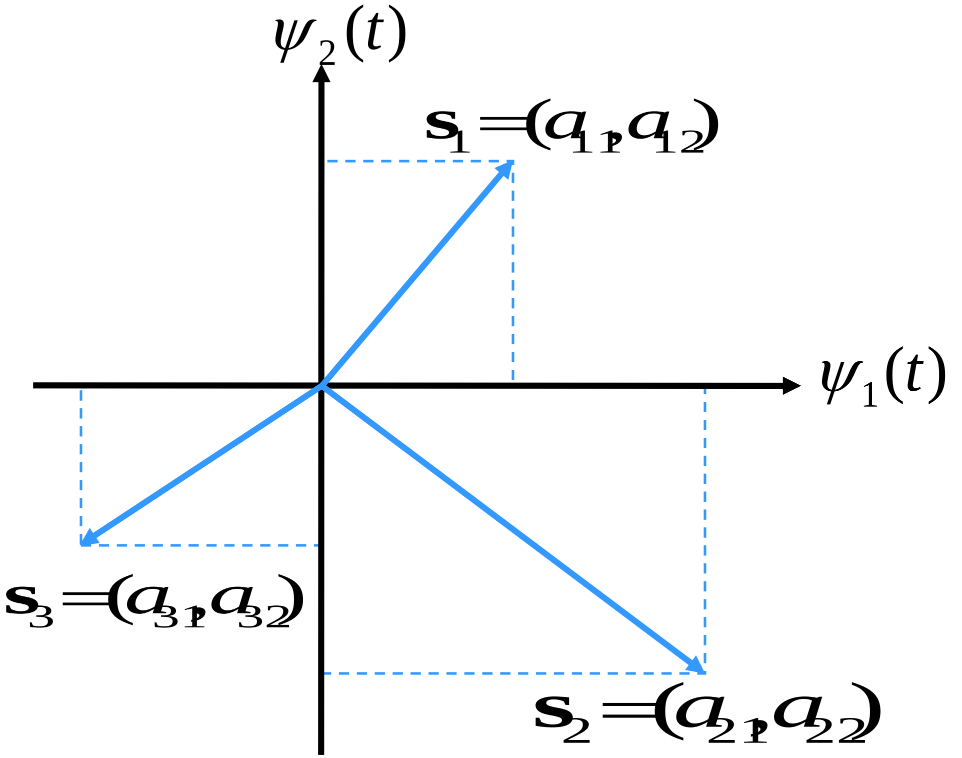
$$s_i(t) = \sum_{j=1}^{N} a_{ij}\psi_j(t)$$

$$\mathbf{s}_i = (a_{i1}, a_{i2}, \ldots, a_{iN})$$

Waveform to vector conversion **Vector to waveform conversion**



Example of projecting signals to an orthonormal signal space



Transmitted signal alternatives

$$
\begin{cases}
s_1(t) = a_{11}\psi_1(t) + a_{12}\psi_2(t) \Leftrightarrow \mathbf{s}_1 = (a_{11}, a_{12}) \\
s_2(t) = a_{21}\psi_1(t) + a_{22}\psi_2(t) \Leftrightarrow \mathbf{s}_2 = (a_{21}, a_{22}) \\
s_3(t) = a_{31}\psi_1(t) + a_{32}\psi_2(t) \Leftrightarrow \mathbf{s}_3 = (a_{31}, a_{32})
\end{cases}
$$

$$a_{ij} = \int_0^T s_i(t)\psi_j(t)dt \qquad j = 1,\ldots,N \qquad i = 1,\ldots,M \qquad 0 \le t \le T$$

Signal space – cont'd

- To find an orthonormal basis functions for a given set of signals, the Gram-Schmidt procedure can be used.
- Gram-Schmidt procedure:
  - Given a signal set $\{s_i(t)\}_{i=1}^{M}$, compute an orthonormal basis $\{\psi_j(t)\}_{j=1}^{N}$

    1. Define $\psi_1(t) = s_1(t)/\sqrt{E_1} = s_1(t)/\|s_1(t)\|$

    2. For $i = 2,\ldots,M$ compute $d_i(t) = s_i(t) - \sum_{j=1}^{i-1} < s_i(t), \psi_j(t) > \psi_j(t)$

       If $d_i(t) \neq 0$ let $\psi_i(t) = d_i(t)/\|d_i(t)\|$

       If $d_i(t) = 0$, do not assign any basis function.

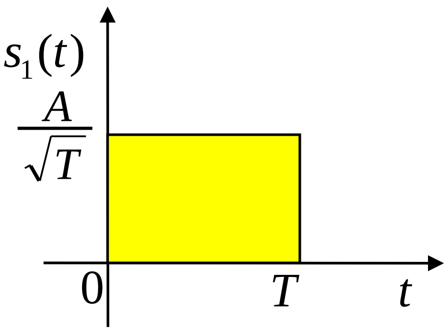
    3. Renumber the basis functions such that basis is

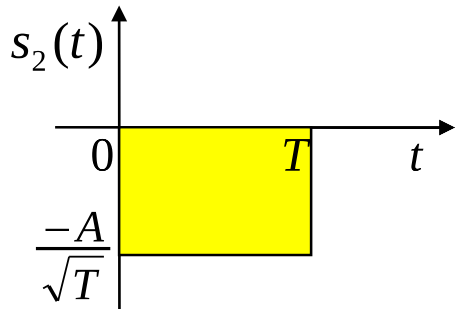
       $\{\psi_1(t), \psi_2(t), \ldots, \psi_N(t)\}$

  - This is only necessary if $d_i(t) = 0$ for any i in step 2.
  - Note that $N \leq M$

Example of Gram-Schmidt procedure

- Find the basis functions and plot the signal space for the following transmitted signals:





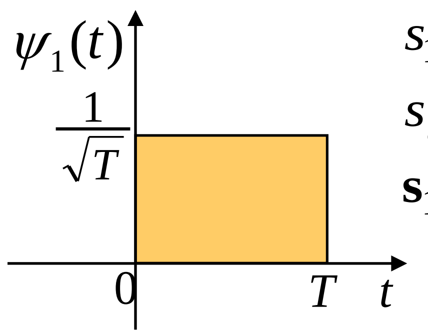
- Using Gram-Schmidt procedure:

1. $$E_1 = \int_0^T |s_1(t)|^2 dt = A^2$$

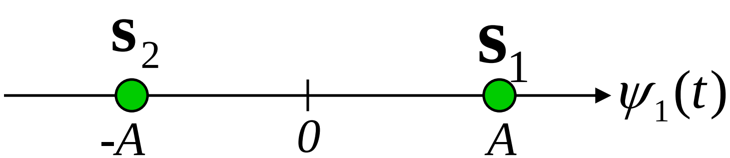
$$\psi_1(t) = s_1(t)/\sqrt{E_1} = s_1(t)/A$$

2. $$< s_2(t), \psi_1(t) >= \int_0^T s_2(t)\psi_1(t)dt = -A$$

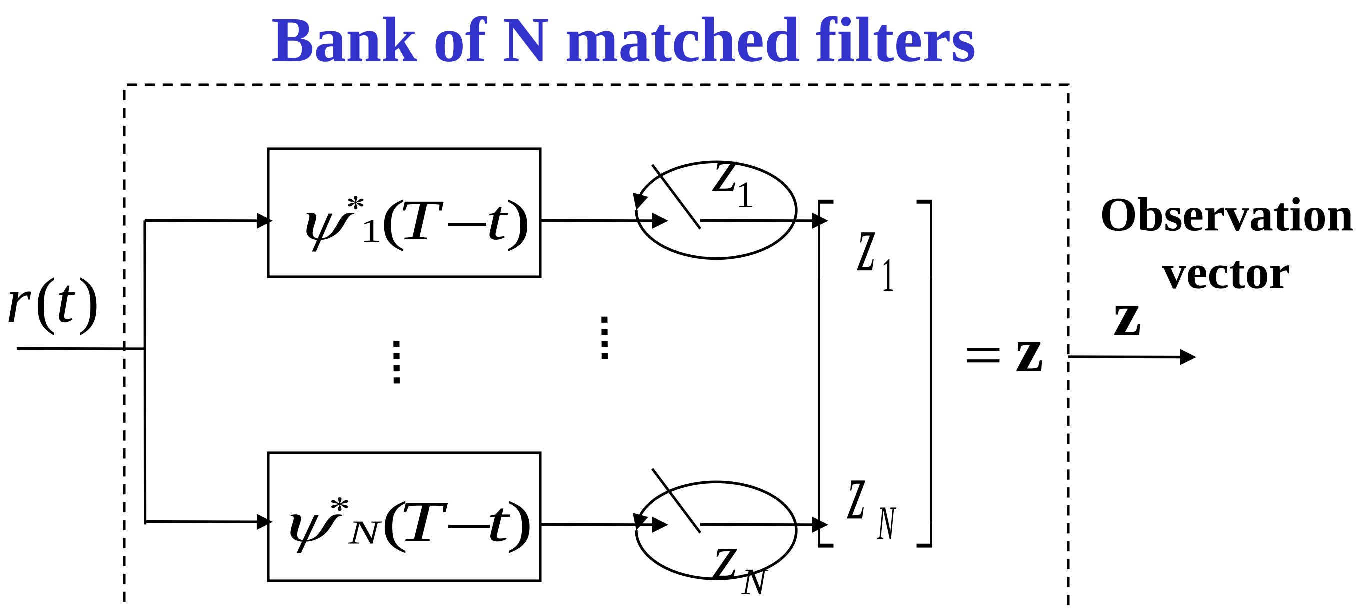
$$d_2(t) = s_2(t) - (-A)\psi_1(t) = 0$$



$$s_1(t) = A\psi_1(t)$$
$$s_2(t) = -A\psi_1(t)$$
$$\mathbf{s}_1 = (A) \quad \mathbf{s}_2 = (-A)$$



Implementation of the matched filter receiver



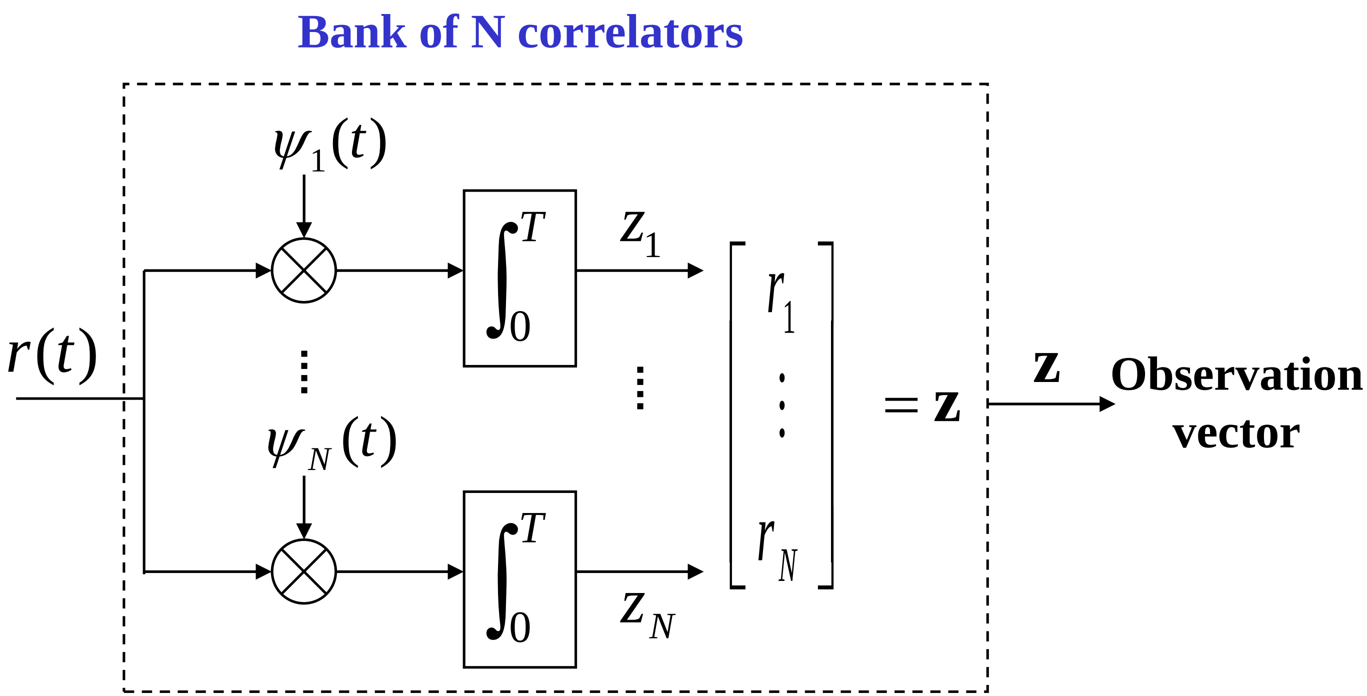
$$s_i(t) = \sum_{j=1}^{N} a_{ij}\psi_j(t) \qquad i = 1,\ldots,M$$

$$\mathbf{z} = (z_1, z_2, \ldots, z_N)$$

$$z_j = r(t) * \psi_j(T-t) \qquad j = 1,\ldots,N$$

$$N \leq M$$

Implementation of the correlator receiver



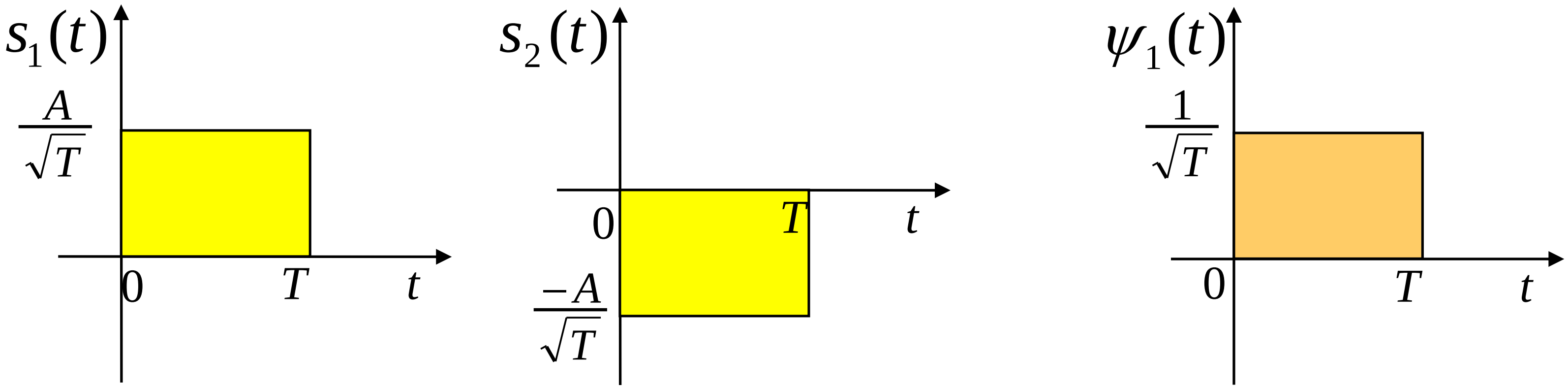
$$s_i(t) = \sum_{j=1}^{N} a_{ij}\psi_j(t) \qquad i = 1,\ldots,M$$

$$\mathbf{z} = (z_1, z_2, \ldots, z_N)$$

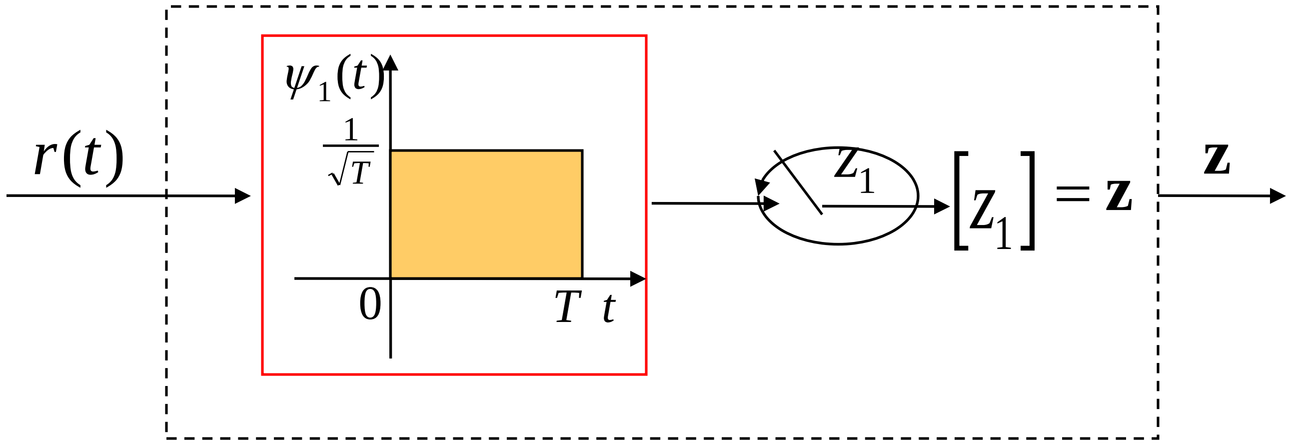
$$N \le M$$

$$z_j = \int_0^T r(t)\psi_j(t)dt \qquad j = 1,\ldots,N$$

Example of matched filter receivers using basic functions



1 matched filter



- Number of matched filters (or correlators) is reduced by 1 compared to using matched filters (correlators) to the transmitted signal.
- Reduced number of filters (or correlators)

White noise in the orthonormal signal space

- AWGN, $n(t)$, can be expressed as

$$n(t) = \hat{n}(t) + \tilde{n}(t)$$

Noise projected on the signal space which impacts the detection process.

Noise outside of the signal space

$$\begin{cases} \hat{n}(t) = \sum_{j=1}^{N} n_j \psi_j(t) \\ n_j = \langle n(t), \psi_j(t) \rangle \quad j = 1, \ldots, N \\ \langle \tilde{n}(t), \psi_j(t) \rangle = 0 \quad j = 1, \ldots, N \end{cases}$$

Vector representation of $\hat{n}(t)$

$$\mathbf{n} = (n_1, n_2, \ldots, n_N)$$

$\{n_j\}_{j=1}^{N}$ independent zero-mean Gaussain random variables with variance $\text{var}(n_j) = N_0/2$