Digital Communications I: Modulation and Coding Course

Spring - 2015

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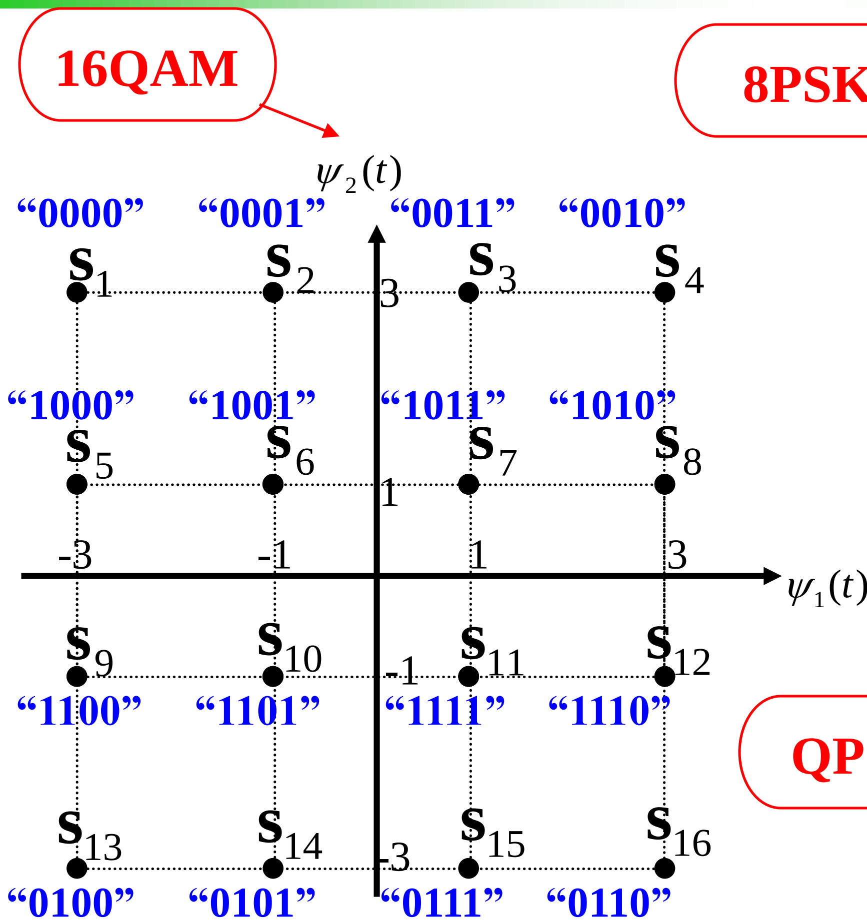
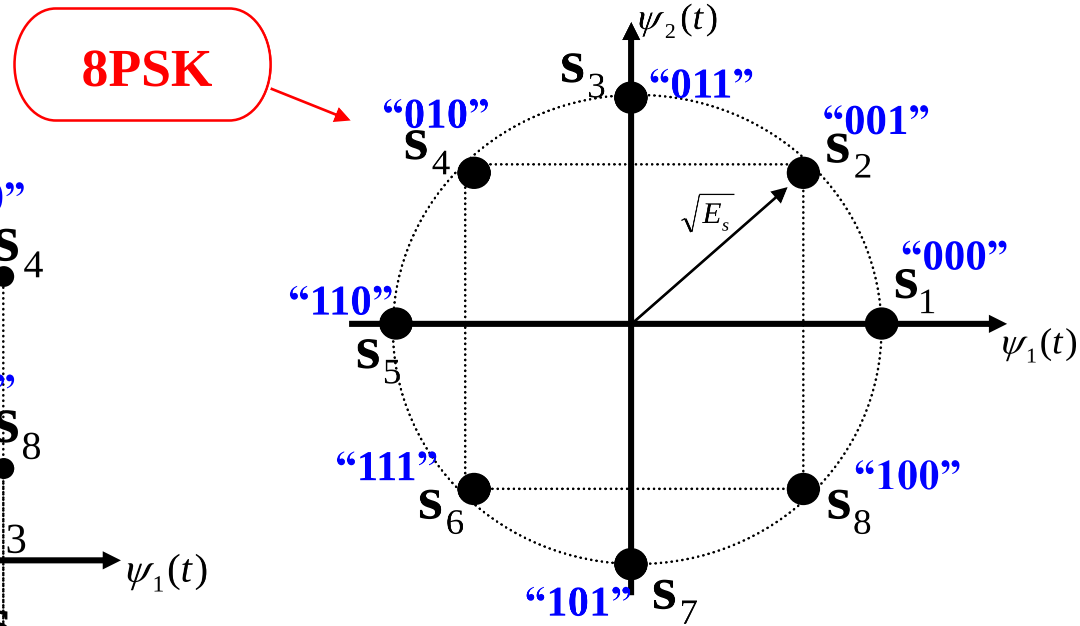
Lecture 4b: Detection of M-ary Bandpass Signals

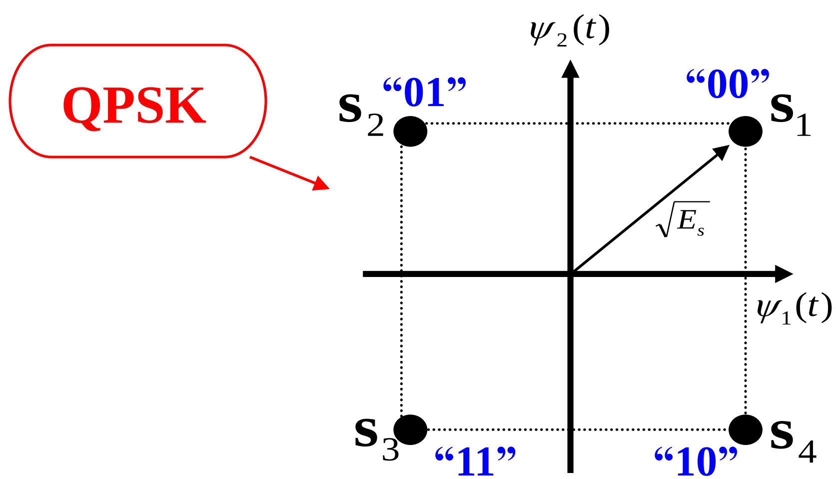
Last time we talked about:

- Some bandpass modulation schemes
  - M-PAM, M-PSK, M-FSK, M-QAM
- How to perform coherent and non-coherent detection

Example of two dim. modulation





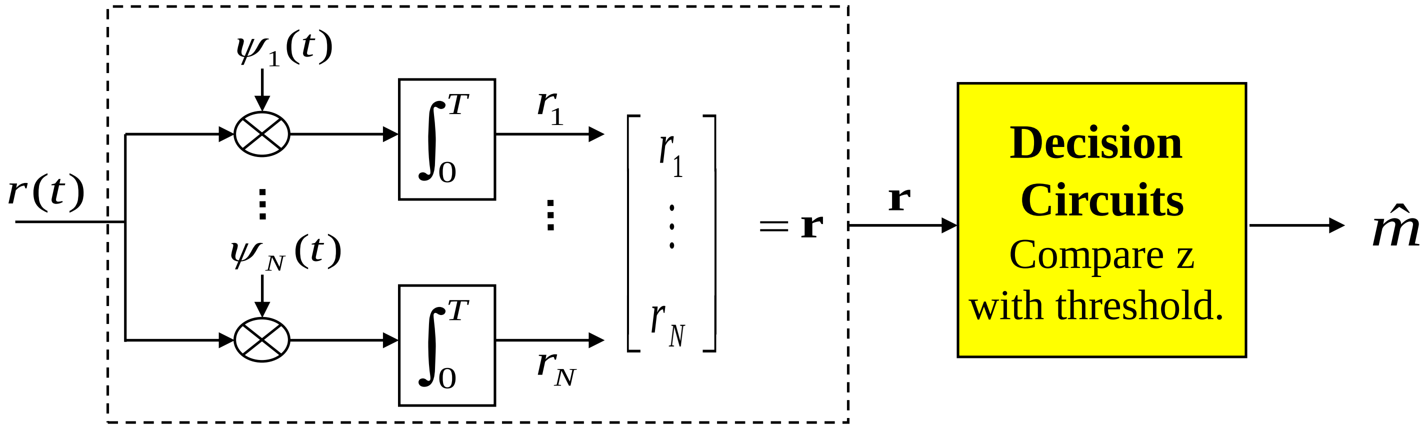


Today, we are going to talk about:

- How to calculate the average probability of symbol error for different modulation schemes that we studied?
- How to compare different modulation schemes based on their error performances?

Error probability of bandpass modulation

- Before evaluating the error probability, it is important to remember that:
  - The type of modulation and detection (coherent or non-coherent) determines the structure of the decision circuits and hence the decision variable, denoted by z.
  - The decision variable, z, is compared with M-1 thresholds, corresponding to M decision regions for detection purposes.



Error probability …

- The matched filters output (observation vector= $\mathbf{r}$) is the detector input and the decision variable is a $z = f(\mathbf{r})$ function of $\mathbf{r}$, i.e.
  - For MPAM, MQAM and MFSK with coherent detection $z = \mathbf{r}$
  - For MPSK with coherent detection $z = \angle\mathbf{r}$
  - For non-coherent detection (M-FSK and DPSK), $z = |\mathbf{r}|$
- We know that for calculating the average probability of symbol error, we need to determine

$$\Pr(\mathbf{r} \text{ lies inside } \mathrm{Z_i} \,|\, \mathbf{s}_i \text{ sent}) \equiv \Pr(\mathrm{z} \text{ satisfies condition } \mathrm{C_i} \,|\, \mathbf{s}_i \text{ sent})$$

- *Hence, we need to know the statistics of z, which depends on the modulation scheme and the detection type.*

Error probability …

- AWGN channel model: $\mathbf{r} = \mathbf{s}_i + \mathbf{n}$
  - The signal vector $\mathbf{s}_i = (a_{i1}, a_{i2}, \ldots, a_{iN})$ is deterministic.
  - The elements of the noise vector $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ are i.i.d Gaussian random variables with zero-mean and variance $N_0/2$. The noise vector's pdf is

$$p_{\mathbf{n}}(\mathbf{n}) = \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{\|\mathbf{n}\|^2}{N_0} \right)$$

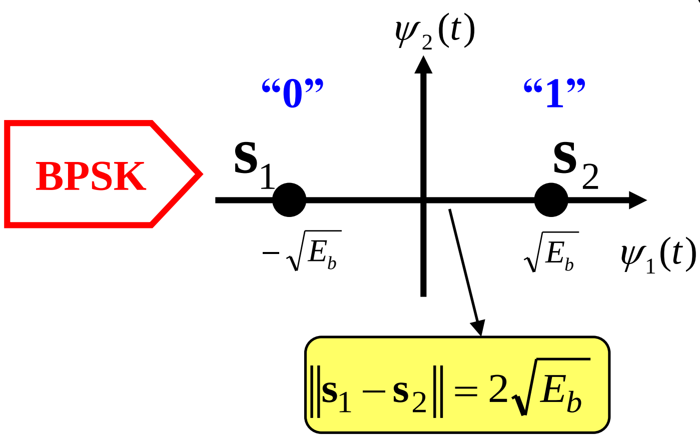
  - The elements of the observed vector $\mathbf{r} = (r_1, r_2, \ldots, r_N)$ are independent Gaussian random variables. Its pdf is

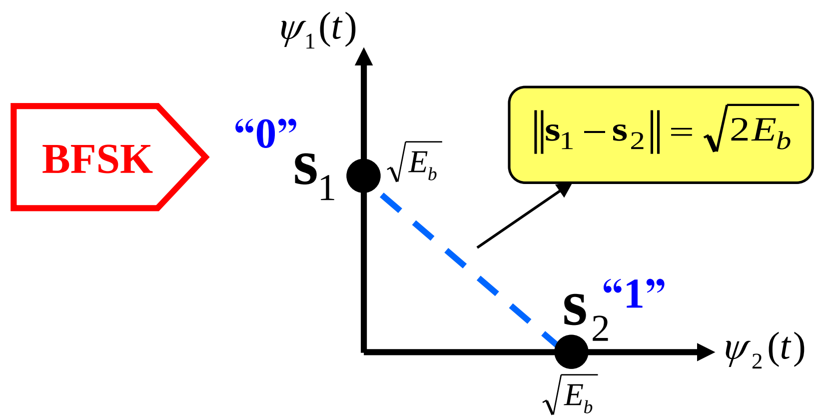
$$p_{\mathbf{r}}(\mathbf{r} \mid \mathbf{s}_i) = \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{\|\mathbf{r} - \mathbf{s}_i\|^2}{N_0} \right)$$

Error probability …

- BPSK and BFSK with *coherent* detection:

$$P_B = Q\left(\frac{\|\mathbf{s}_1 - \mathbf{s}_2\|/2}{\sqrt{N_0/2}}\right)$$





$$P_B = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

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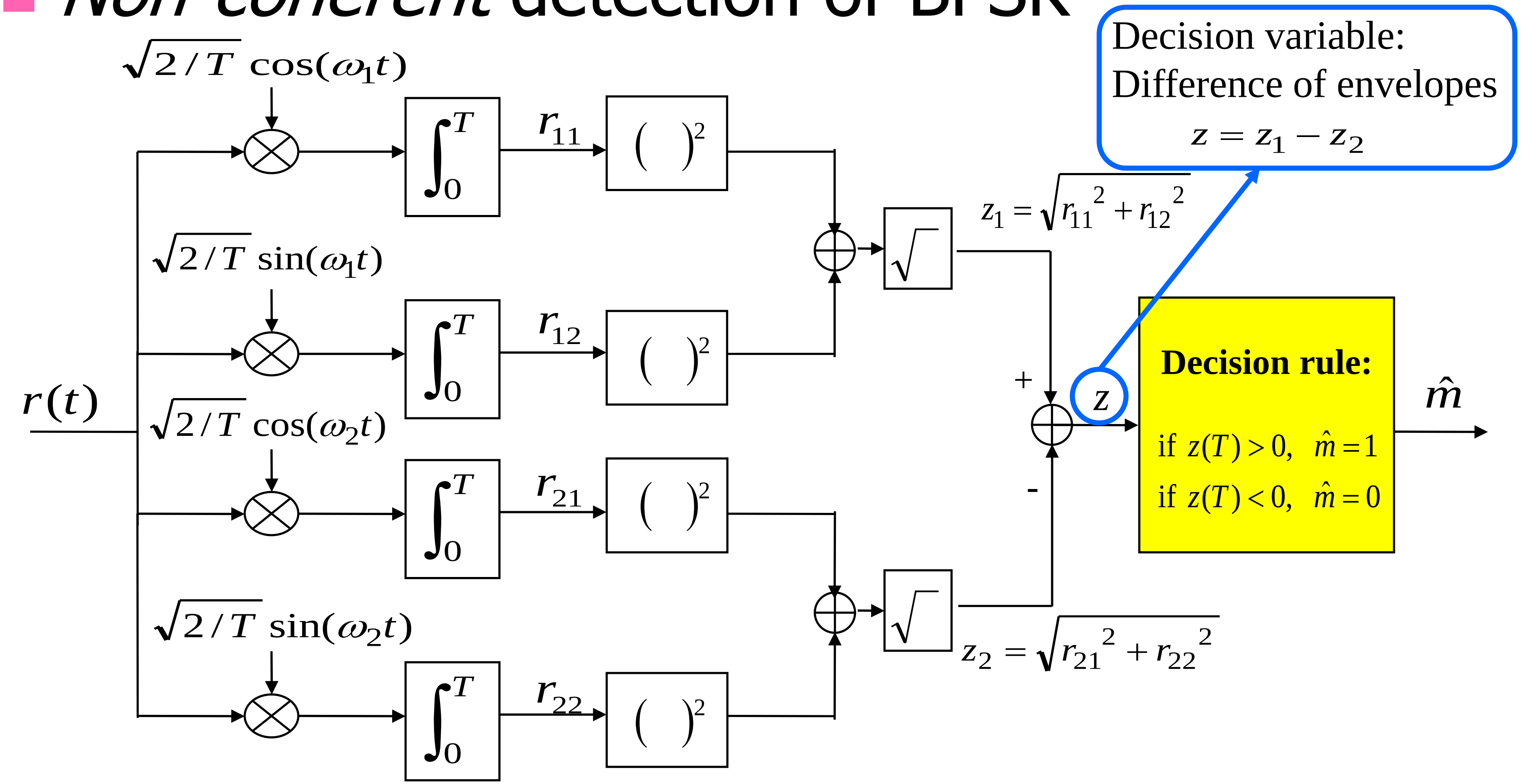
Error probability …

- *Non-coherent* detection of BFSK



Error probability – cont'd

- *Non-coherent* detection of BFSK ...

$$P_B = \frac{1}{2}\Pr(z_1 > z_2 \mid \mathbf{s}_2) + \frac{1}{2}\Pr(z_2 > z_1 \mid \mathbf{s}_1)$$

$$= \Pr(z_1 > z_2 \mid \mathbf{s}_2) = E\left[\Pr(z_1 > z_2 \mid \mathbf{s}_2, z_2)\right]$$

$$= \int_0^\infty \Pr(z_1 > z_2 \mid \mathbf{s}_2, z_2)\, p(z_2 \mid \mathbf{s}_2) dz_2 = \int_0^\infty \left[\int_{z_2}^\infty p(z_1 \mid \mathbf{s}_2) dz_1\right] p(z_2 \mid \mathbf{s}_2) dz_2$$

$p(z_1 \mid \mathbf{s}_2)$: Rayleigh pdf

$p(z_2 \mid \mathbf{s}_2)$: Rician pdf

$$P_B = \frac{1}{2}\exp\left(-\frac{E_b}{2N_0}\right)$$

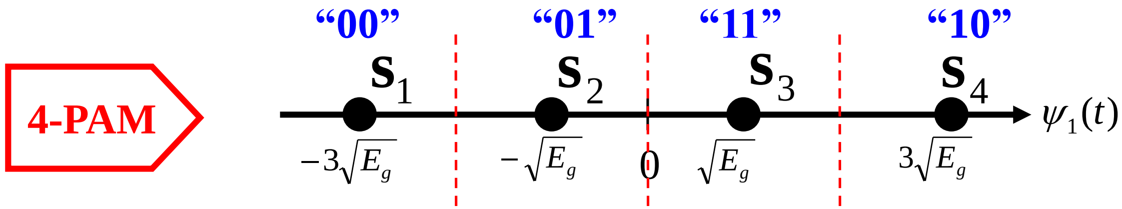
- Similarly, *non-coherent* detection of DBPSK

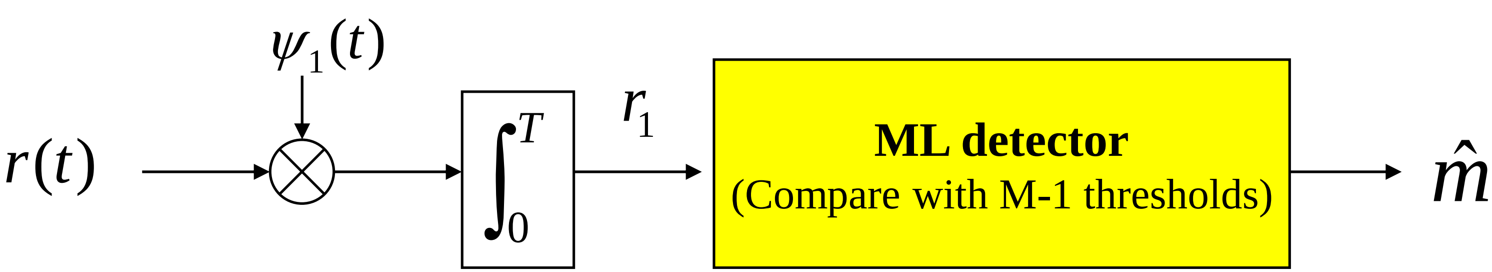
$$P_B = \frac{1}{2}\exp\left(-\frac{E_b}{N_0}\right)$$

Error probability ….

- *Coherent* detection of M-PAM
  - Decision variable:

$$z = r_1$$





Error probability ….

- *Coherent* detection of M-PAM ….
  - Error happens if the noise, $n_1 = r_1 - \mathbf{s}_m$, exceeds in amplitude one-half of the distance between adjacent symbols. For symbols on the border, error can happen only in one direction. Hence:

$$P_e(\mathbf{s}_m) = \Pr\left(|n_1| = |r_1 - \mathbf{s}_m| > \sqrt{E_g}\right) \quad \text{for} \quad 2 < m < M-1;$$

$$P_e(\mathbf{s}_1) = \Pr\left(n_1 = r_1 - \mathbf{s}_1 > \sqrt{E_g}\right) \quad \text{and} \quad P_e(\mathbf{s}_M) = \Pr\left(n_1 = r_1 - \mathbf{s}_M < -\sqrt{E_g}\right)$$

$$P_E(M) = \frac{1}{M}\sum_{m=1}^{M} P_e(\mathbf{s}_m) = \frac{M-2}{M}\Pr\left(|n_1| > \sqrt{E_g}\right) + \frac{1}{\mathrm{M}}\Pr\left(n_1 > \sqrt{E_g}\right) + \frac{1}{\mathrm{M}}\Pr\left(n_1 < -\sqrt{E_g}\right)$$

$$= \frac{2(M-1)}{M}\Pr\left(n_1 > \sqrt{E_g}\right) = \frac{2(M-1)}{M}\int_{\sqrt{E_g}}^{\infty} p_{n_1}(n)\,dn = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{2E_g}{N_0}}\right)$$

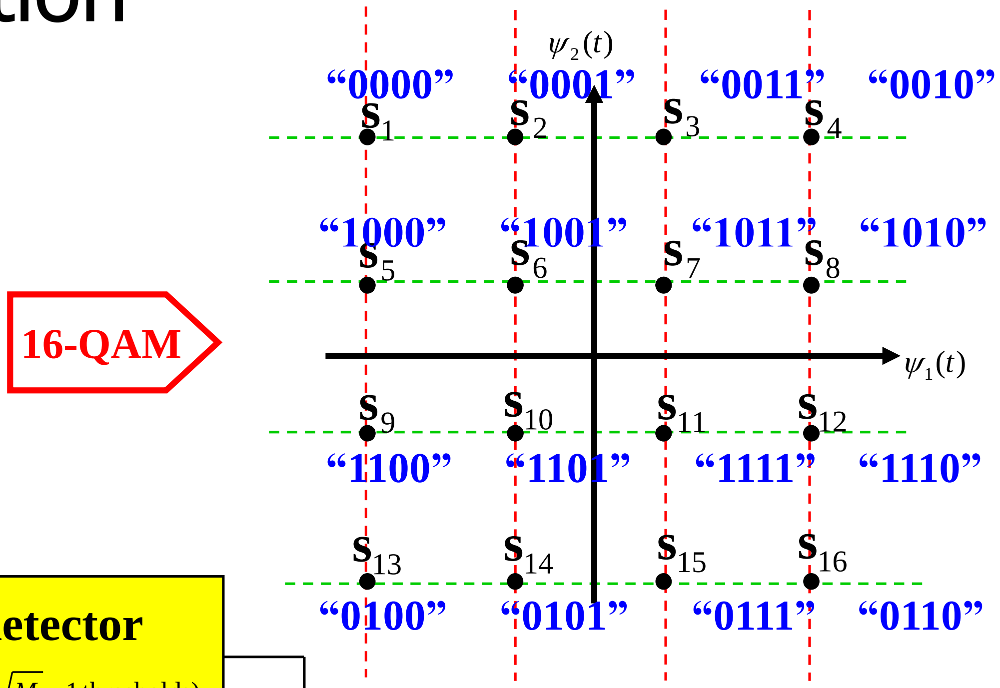
$$E_s = (\log_2 M) E_b = \frac{(M^2-1)}{3} E_g$$

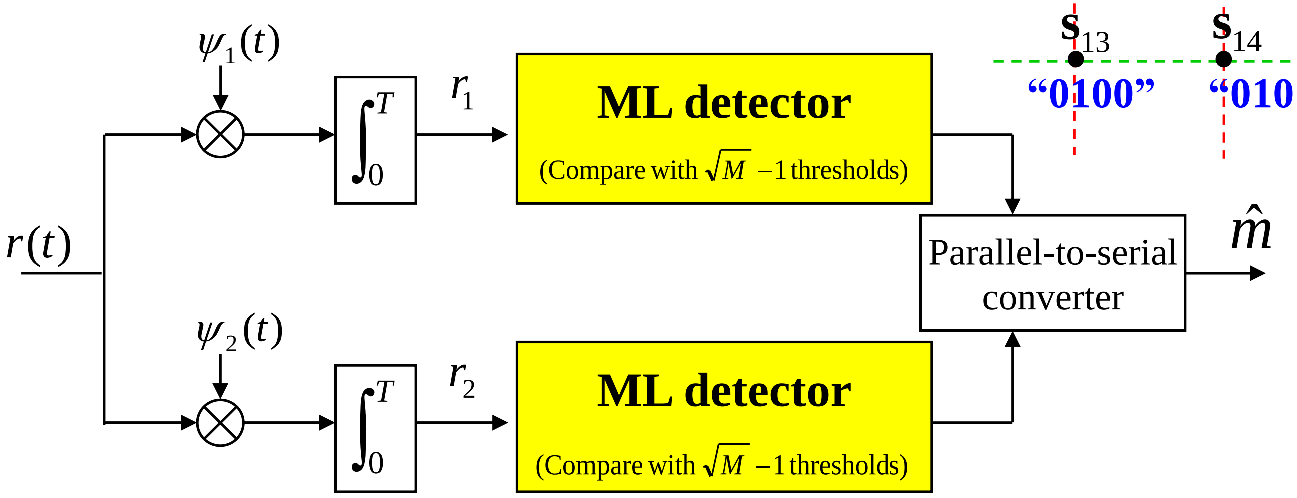
Gaussian pdf with zero mean and variance $N_0/2$

$$P_E(M) = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6\log_2 M}{M^2-1}\frac{E_b}{N_0}}\right)$$

Error probability …

- *Coherent* detection of M-QAM





Error probability …

- *Coherent* detection of M-QAM …
  - M-QAM can be viewed as the combination of two $\sqrt{M}-\mathrm{PAM}$ modulations on I and Q branches, respectively.
  - No error occurs if no error is detected on either the I or the Q branch.
  - Considering the symmetry of the signal space and the orthogonality of the I and Q branches:

$$P_E(M)=1-P_C(M)=1-\Pr(\text{no error detected on I and Q branches})$$

$$\Pr(\text{no error detected on I and Q branches})=\Pr(\text{no error on I})\Pr(\text{no error on Q})$$

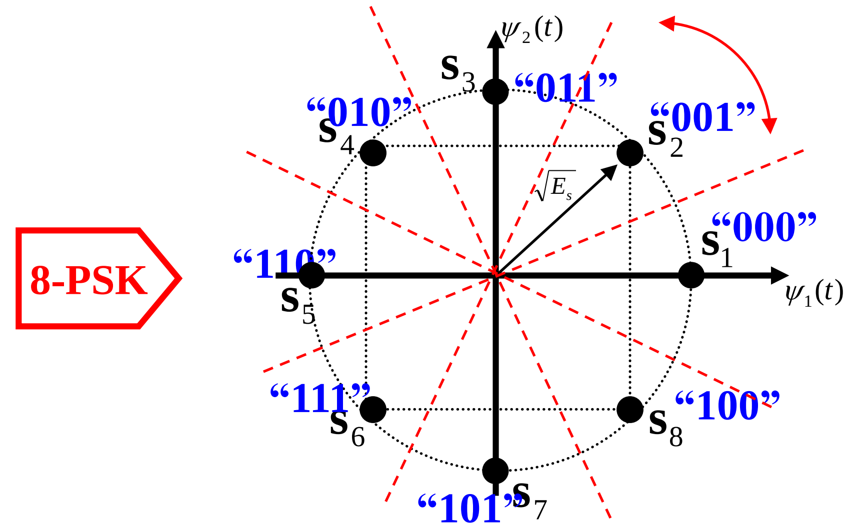
$$=\Pr(\text{no error on I})^2=\left(1-P_E\left(\sqrt{M}\right)\right)^2$$

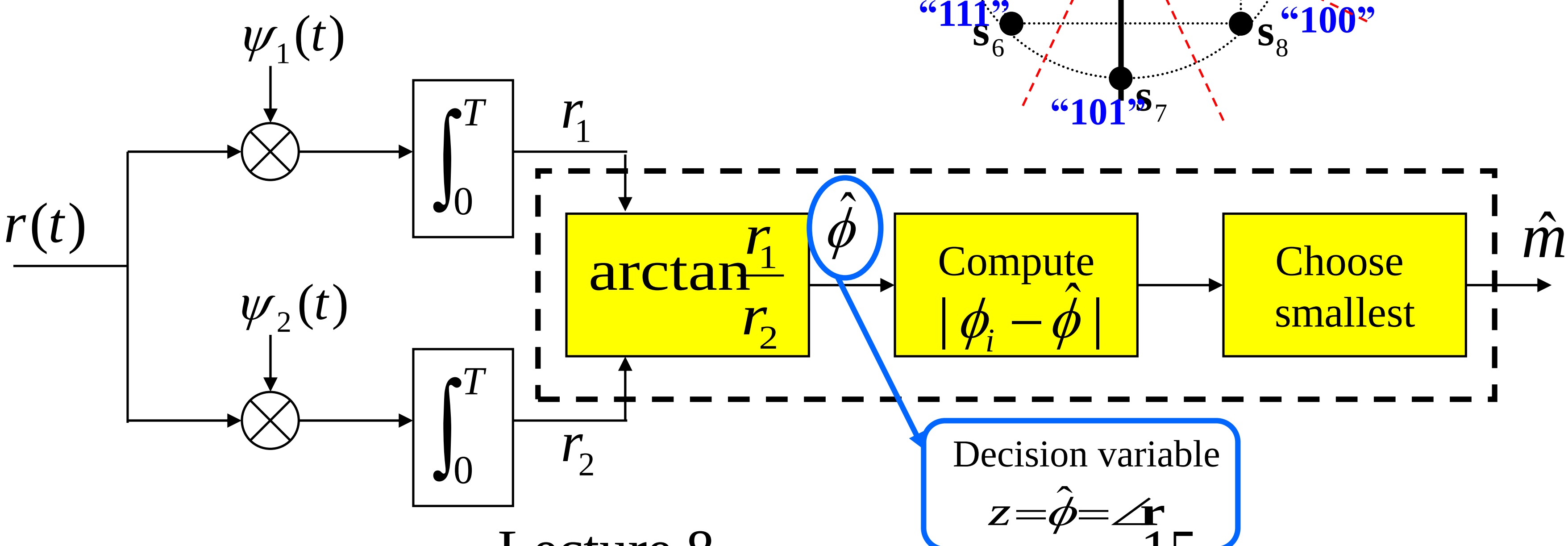
Average probability of symbol error for $\sqrt{M}-\mathrm{PAM}$

$$P_E(M)=4\left(1-\frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{3\log_2 M}{M-1}\frac{E_b}{N_0}}\right)$$

Error probability …

- *Coherent* detection of MPSK





Error probability …

- *Coherent* detection of MPSK …
  - The detector compares the phase of observation vector to M-1 thresholds.
  - Due to the circular symmetry of the signal space, we have:

$$P_E(M) = 1 - P_C(M) = 1 - \frac{1}{M}\sum_{m=1}^{M} P_c(\mathbf{s}_m) = 1 - P_c(\mathbf{s}_1) = 1 - \int_{-\pi/M}^{\pi/M} p_{\hat{\phi}}(\phi)\, d\phi$$

where

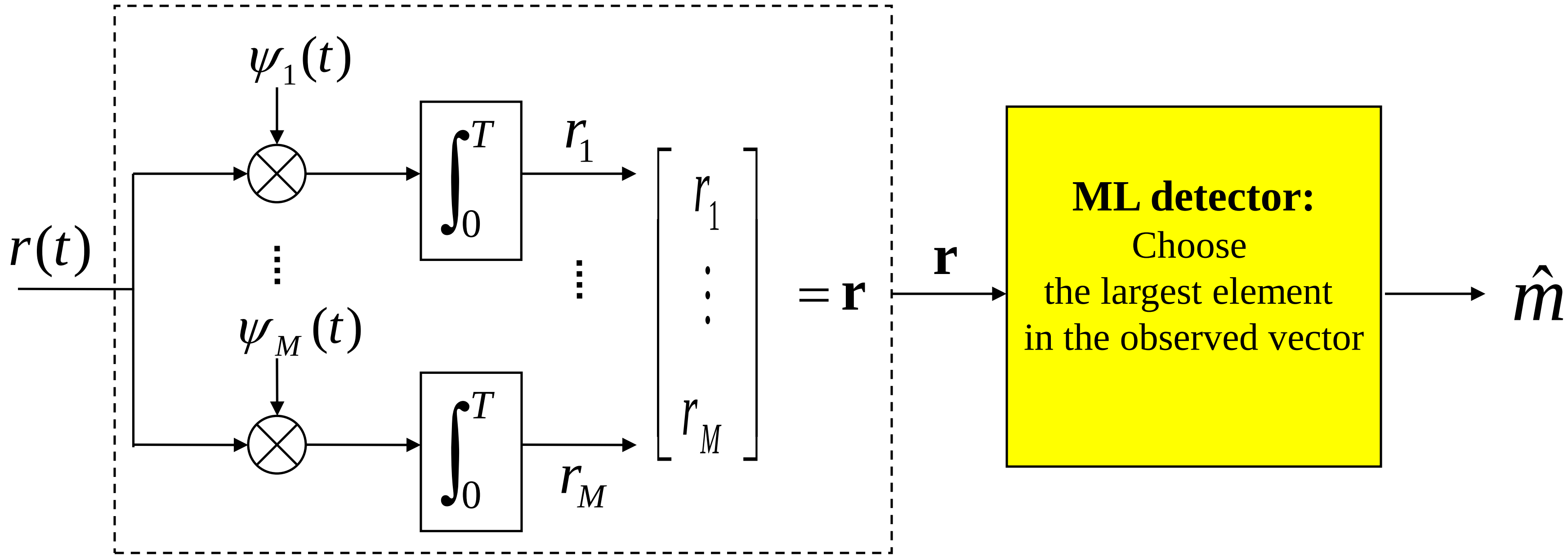
$$p_{\hat{\phi}}(\phi) \approx \sqrt{\frac{2}{\pi}\frac{E_s}{N_0}} \cos(\phi) \exp\left(-\frac{E_s}{N_0}\sin^2\phi\right); \quad |\phi| \le \frac{\pi}{2}$$

  - It can be shown that

$$P_E(M) \approx 2Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\left(\frac{\pi}{M}\right)\right) \text{ or } P_E(M) \approx 2Q\left(\sqrt{\frac{2(\log_2 M)E_b}{N_0}}\sin\left(\frac{\pi}{M}\right)\right)$$

Error probability …

- *Coherent* detection of M-FSK



Error probability …

- *Coherent* detection of M-FSK …
- The dimension of the signal space is *M*. An upper bound for the average symbol error probability can be obtained by using the union bound. Hence:

$$P_E(M) \le (M-1)Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

or, equivalently

$$P_E(M) \le (M-1)Q\left(\sqrt{\frac{(\log_2 M)E_b}{N_0}}\right)$$

Bit error probability versus symbol error probability

- Number of bits per symbol $k = \log_2 M$
- For orthogonal M-ary signaling (M-FSK)

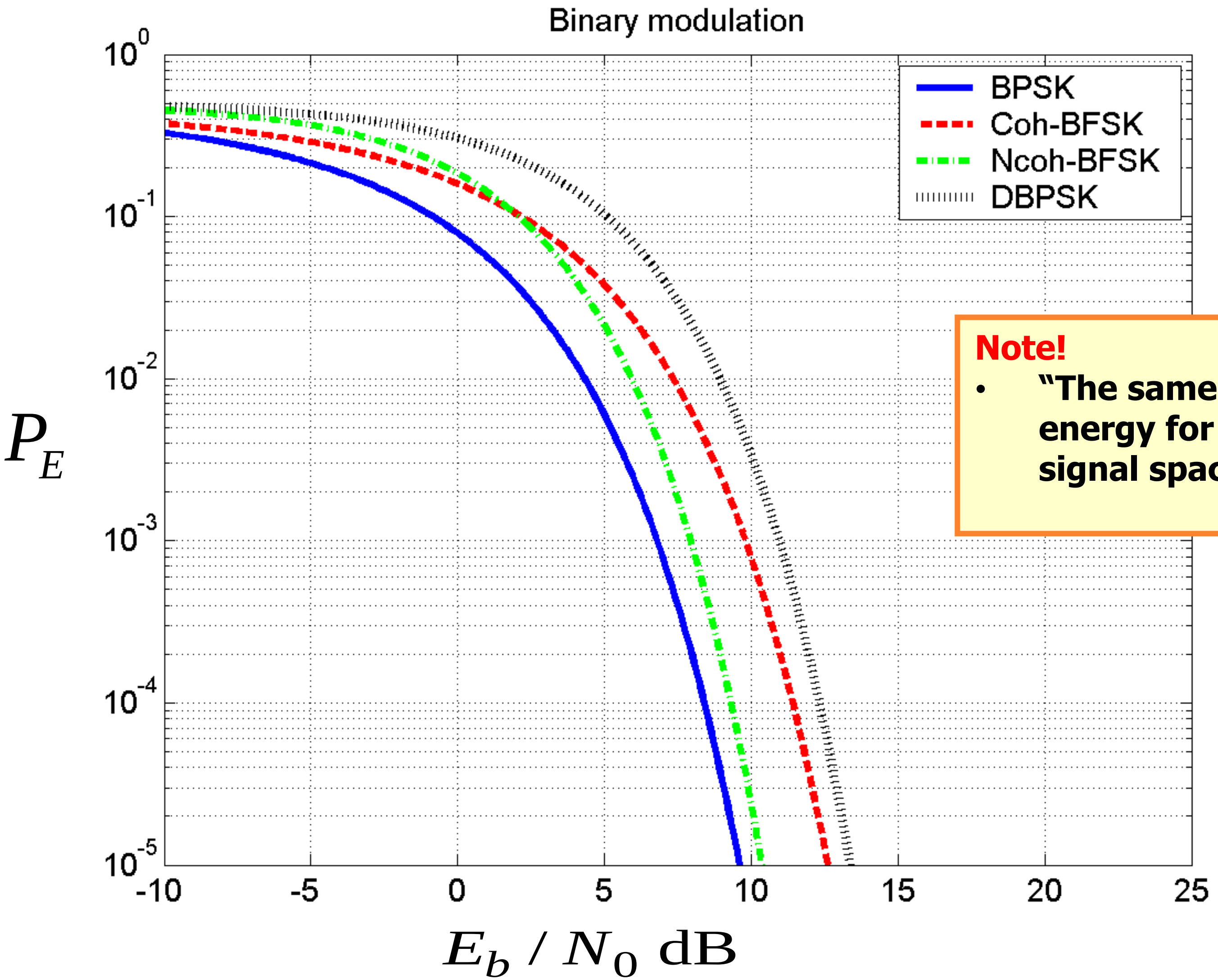
$$\frac{P_B}{P_E} = \frac{2^{k-1}}{2^k - 1} = \frac{M/2}{M-1}$$

$$\lim_{k \to \infty} \frac{P_B}{P_E} = \frac{1}{2}$$

- For M-PSK, M-PAM and M-QAM

$$P_B \approx \frac{P_E}{k} \quad \text{for } P_E << 1$$

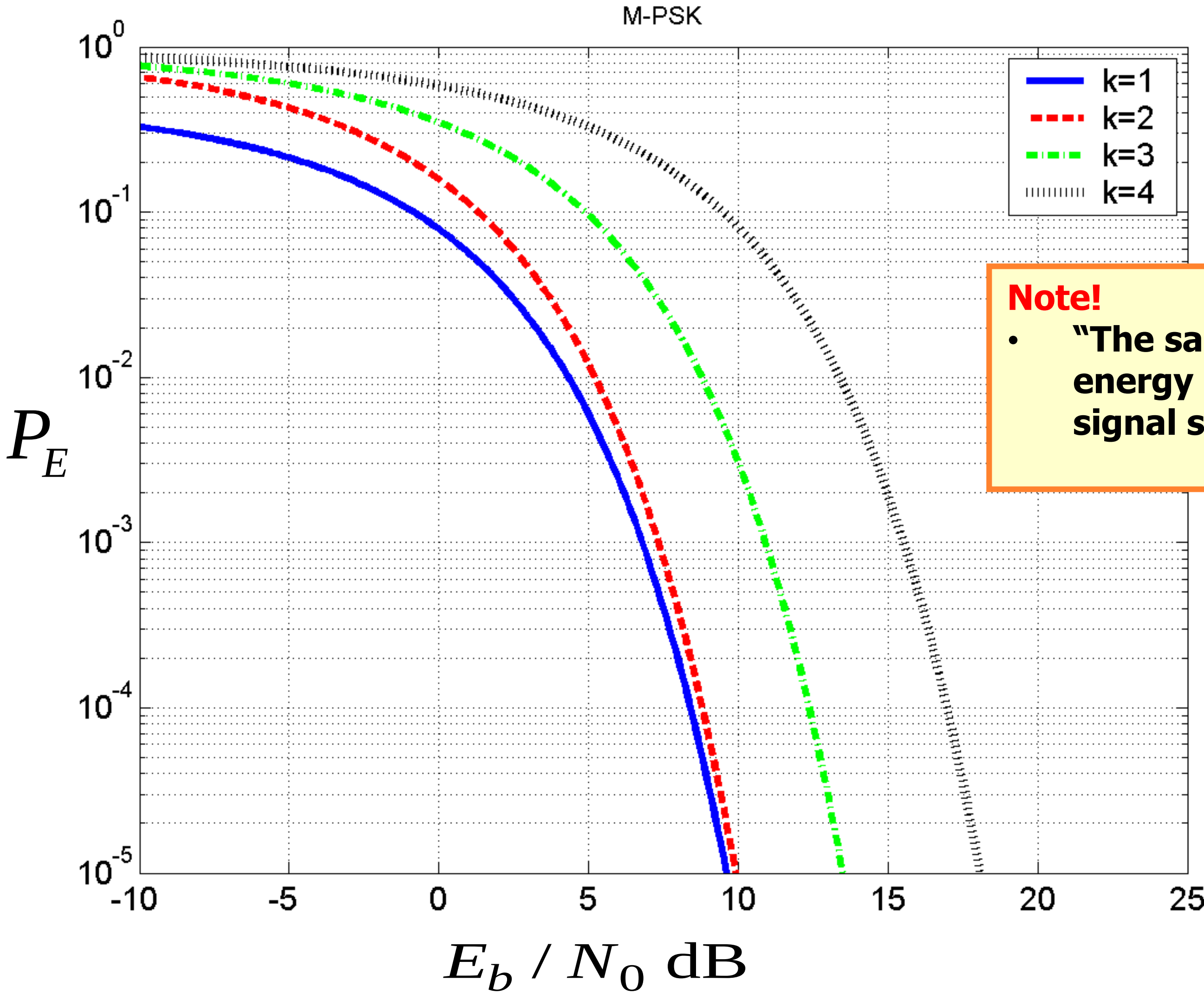
Probability of symbol error for binary modulation



Note!

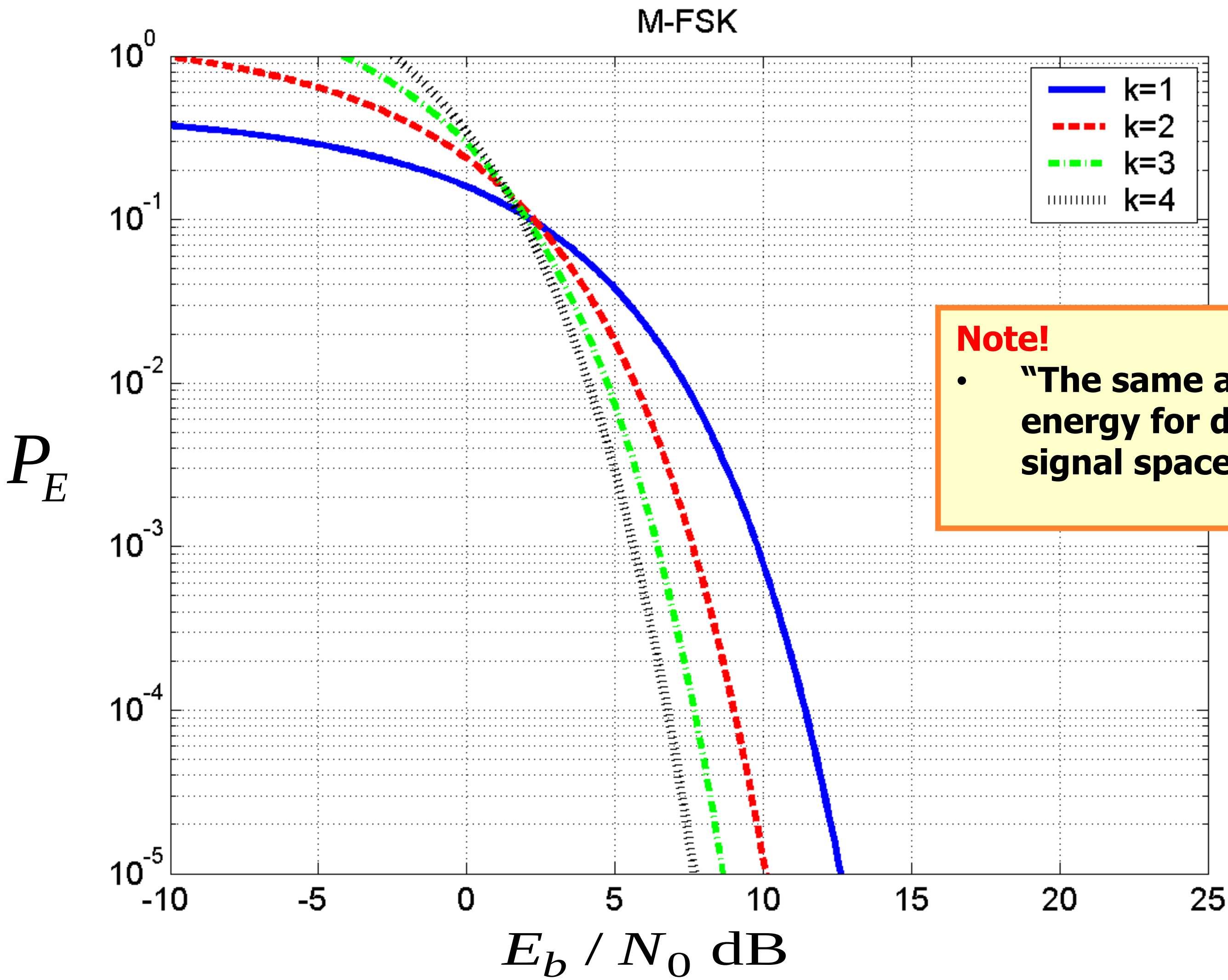
- **"The same average symbol energy for different sizes of signal space"**

Probability of symbol error for M-PSK



Note!
- **"The same average symbol energy for different sizes of signal space"**

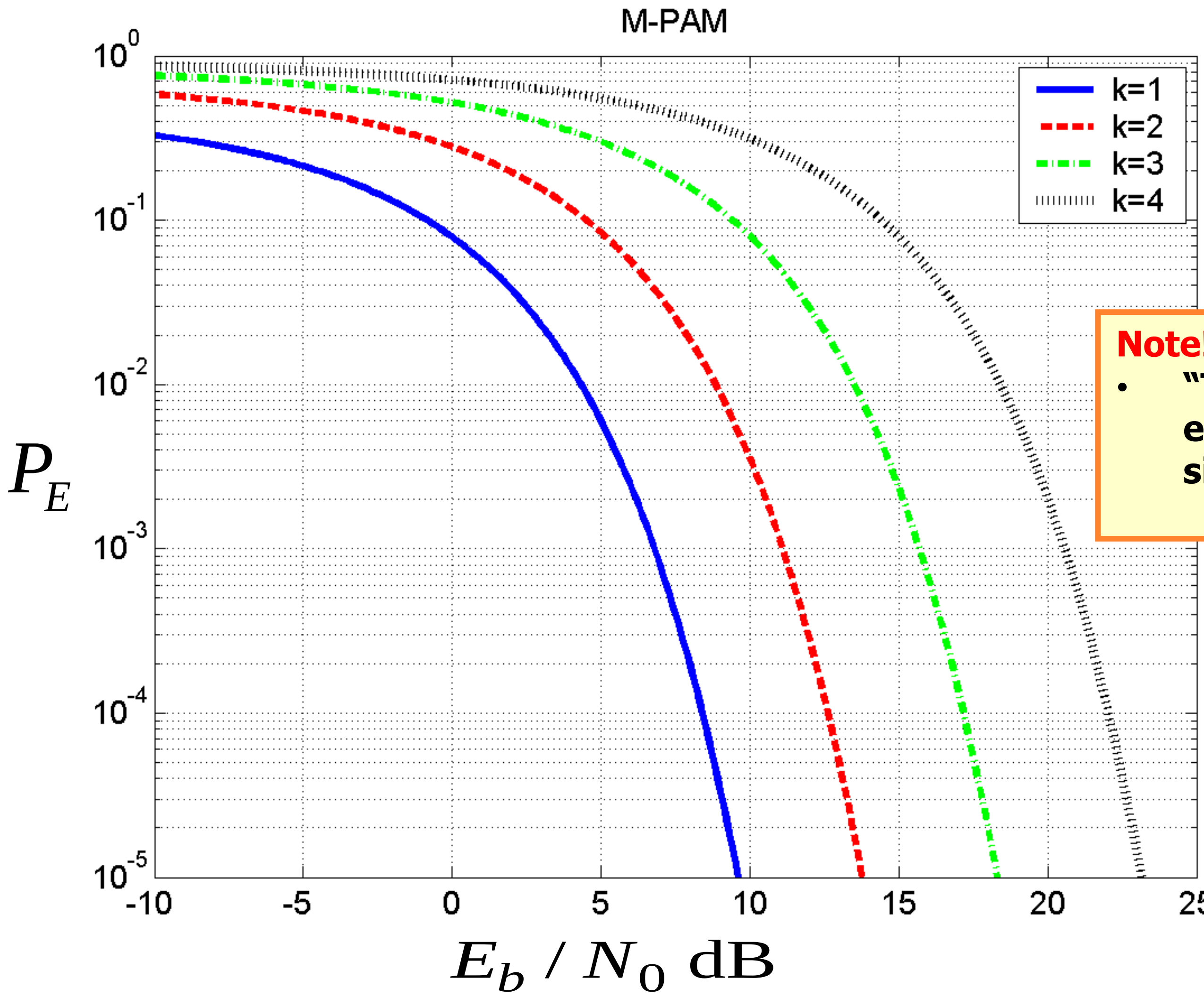
Probability of symbol error for M-FSK



Note!

- **"The same average symbol energy for different sizes of signal space"**

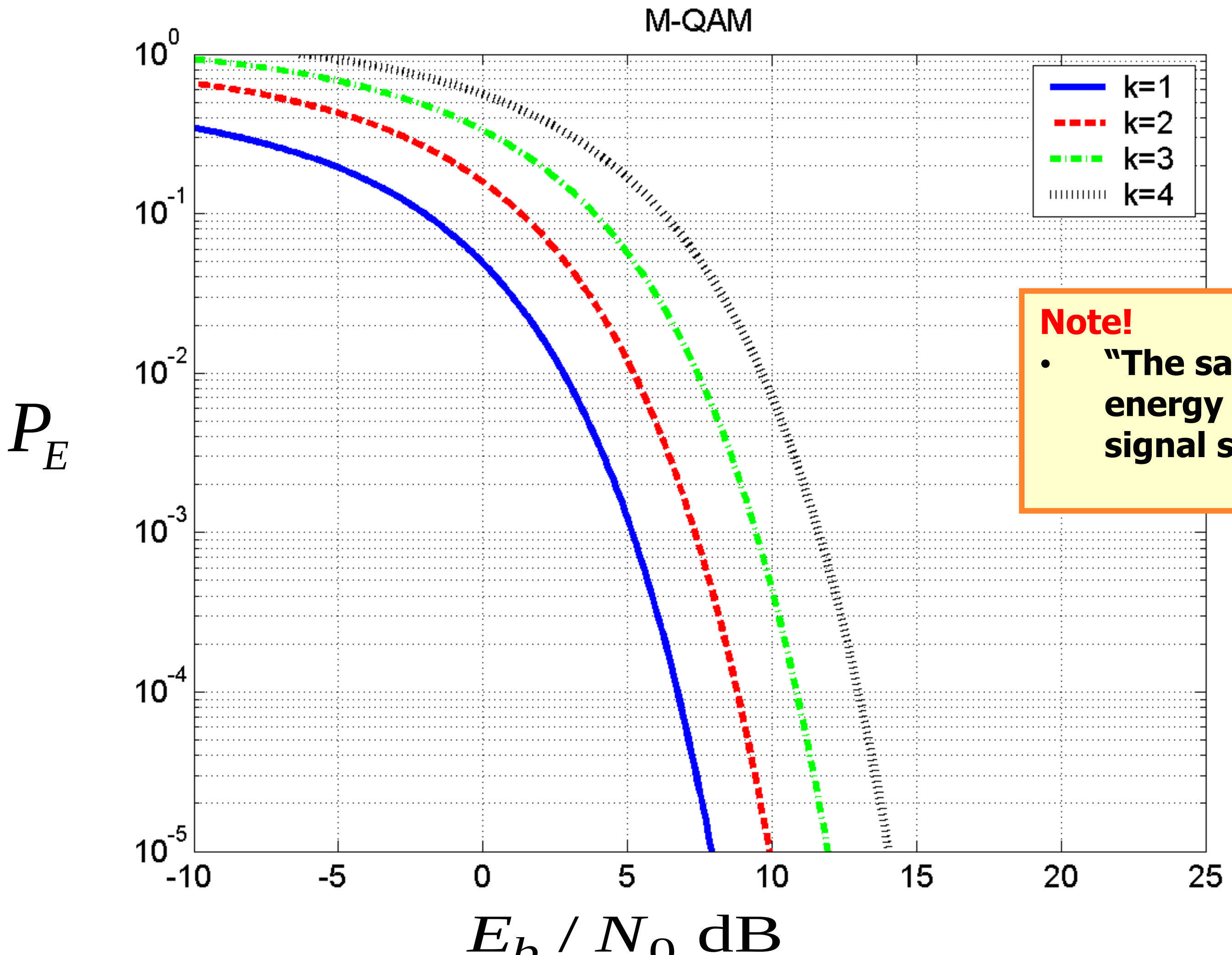
Probability of symbol error for M-PAM



Note!

- **"The same average symbol energy for different sizes of signal space"**

Probability of symbol error for M-QAM



Note!

- **"The same average symbol energy for different sizes of signal space"**

Example of samples of matched filter output for some bandpass modulation schemes