EE 3025 S2005 Homework Set #10

(due 10:10 AM Friday, April 22, 2005)

Directions: Work all 5 problems. We will grade Problem 1 and will randomly choose two of the other problems for grading.

1. Let $Z_n,\ n = 0, \pm 1, \pm 2, \pm 3, \cdots$ be a bilateral Gaussian white noise process with variance one. (This means that the Z_n's are independent random variables, each Gaussian with mean zero and variance one.) Suppose we filter the random signal Z_n using a linear time-invariant filter with transfer function $H(z) = a + bz^{-1} + cz^{-2}$. Let X_n be the random signal produced at the filter output:

$$Z_n \rightarrow \boxed{H(z)} \rightarrow X_n$$

This means that for every integer time n, we have

$$X_n = aZ_n + bZ_{n-1} + cZ_{n-2}.$$

(This is a special case of what is called FIR filtering in EE 3015.) I will prove in class that the autocorrelation function $R_X(\tau)$ satisfies

$$\begin{aligned} R_X(0) &= a^2 + b^2 + c^2 \\ R_X(\pm 1) &= ab + bc \\ R_X(\pm 2) &= ac \\ R_X(\tau) &= 0, \ \ otherwise \end{aligned}$$

In this Matlab problem, you are going to fulfill two goals:

Design: You are going to design the filter by finding the filter tap weights a, b, c that are needed for certain autocorrelation properties of the X process to hold.

Simulation: You are going to make sure the filter you designed is operating correctly by estimating the autocorrelations you get when you use the filter to filter Gaussian white noise samples.

(a) This is the design part of the problem. Use Matlab function "`solve`" to find filter tap weights a, b, c so that

$$\begin{aligned} R_X(0) &= 8 \\ R_X(\pm 1) &= -4 \\ R_X(\pm 2) &= 1 \end{aligned}$$

(There might be more than one solution for a, b, c. Be sure to pick a solution for which a, b, c are real numbers.)

(b) This is the simulation part of the problem. Use Matlab to simulate enough Gaussian white noise samples Z_n so that when you filter them, you obtain a simulation of one realization of X_n for times $n = 1, 2, 3, \cdots, 50000$. Do time-averaging along this one piece of realization to find estimates of each of the three autocorrelations $R_X(0)$, $R_X(1)$, $R_X(2)$ to see how close your estimates are to $8, -4, 1$, respectively. Turn in printout of your Matlab script for part(b) and printout of the result of your program run. (Hint: There will be an experiment in Recitation 12 where you use time-averaging to estimate autocorrelation figures; use that experiment as a guide.)

2. Let X_n be a discrete-time WSS process with autocorrelation function as in Problem 1:

$$\begin{aligned} R_X(0) &= 8 \\ R_X(\pm 1) &= -4 \\ R_X(\pm 2) &= 1 \\ R_X(\tau) &= 0, \quad \textit{otherwise} \end{aligned}$$

(a) Use the orthogonality principle to find the constant A such that the best mean-square first order predictor of X_n will be

$$\hat{X}_n = AX_{n-1}.$$

(In other words, the mean-square prediction error $E[(X_n - \hat{X}_n)^2]$ will be minimized for your first-order predictor.)

(b) Use the orthogonality principle to find the constants B_1 and B_2 such that the best mean-square second order predictor of X_n will be

$$\hat{X}_n = B_1 X_{n-1} + B_2 X_{n-2}.$$

(In other words, the mean-square prediction error $E[(X_n - \hat{X}_n)^2]$ will be minimized for your second-order predictor.)

(c) Use the orthogonality principle to find the constants C_1, C_2, and C_3 such that the best mean-square third order predictor of X_n will be

$$\hat{X}_n = C_1 X_{n-1} + C_2 X_{n-2} + C_3 X_{n-3}.$$

(In other words, the mean-square prediction error $E[(X_n - \hat{X}_n)^2]$ will be minimized for your third-order predictor.)

(d) The decibel figure for the mean square prediction error of a predictor $\hat{X}_n$ used to predict X_n is given by

$$10 \log_{10} \left\{ \frac{E[X_n^2]}{E[(X_n - \hat{X}_n)^2]} \right\}. \tag{1}$$

(Note: This figure will not depend on n because the process is WSS.) Via Matlab, use 50000 simulated X_n values and the corresponding 50000 simulated predictor values $\hat{X}_n$ to estimate the decibel figure (1) for

- Your first-order predictor from (a);
- Your second-order predictor from (b);
- Your third-order predictor from (c).

(Note: The decibel figures should *become bigger* as you go from (a) to (c).) Turn in your Matlab script and printout of its run.

3. Let $X(t)$, $t \geq 0$ be the Poisson process with arrival rate of $\lambda = 1$ arrival per second.

(a) Compute the probability that exactly one more customer arrives in the time interval $[1, 2)$ than in the time interval $[0, 1)$. (Hint: Letting $U = X(1)$ and $V = X(2) - X(1)$, U and V are independent random variables which give the number of arrivals in the time intervals $[0, 1)$ and $[1, 2)$, respectively. You are to compute the probability $P[V = U + 1]$, which can be broken down as $\sum_{k=0}^{\infty} P[U = k, V = k + 1]$. This infinite series can be summed via Matlab.)

(b) Compute the probability that exactly two more customers arrive in the time interval $[1, 2)$ than in the time interval $[0, 1)$.

4. Let $f(t)$ be an arbitrary (deterministic) periodic signal with period 1. Let U be uniformly distributed in the interval $[0, 1]$. Then we can define a random process as follows:

$$X(t) = f(t + U), \quad -\infty < t < \infty.$$

This process is WSS and has autocorrelation function

$$R_X(\tau) = \int_0^1 f(t) f(t + \tau) dt.$$

(I will prove the preceding formula in class.) In each part of the problem, find an analytic expression for $R_X(\tau)$ and then also turn in a Matlab plot which plots $R_X(\tau)$ over the interval $-1/2 \leq \tau \leq 1/2$. (You will find that $R_X(\tau)$ is periodic with period 1, so your plot of $R_X(\tau)$ for $-1/2 \leq \tau \leq 1/2$ will be sufficient.)

(a) The periodic signal $f(t)$ is equal to 1 for $0 \leq t \leq 1/2$ and is equal to -1 for $1/2 < t < 1$.

(b) The periodic signal $f(t)$ is equal to t for $0 \leq t < 1$.

5. (**Note: Do not do this problem for Homework 10. Just do the preceding four problems for Homework 10. I will instead put this problem on Homework 11.**) Let $X(t)$ be a WSS Gaussian random process with mean -4 and autocorrelation function $R_X(\tau) = 16 + 5\exp(-|\tau|)$. Let $Y(t)$ be a WSS Gaussian random process independent of $X(t)$ with mean 5 and autocorrelation function $R_Y(\tau) = 25 + 20\exp(-|\tau|)$.

(a) The process

$$Z_1(t) = X(t)Y(t)$$

is WSS. Find the autocorrelation function $R_{Z_1}(\tau)$. Find the power generated by $Z_1(t)$.

(b) The process

$$Z_2(t) = X(t) + Y(t)$$

is WSS. Find the autocorrelation function $R_{Z_2}(\tau)$. Find the power generated by $Z_2(t)$.

(c) The process

$$Z_3(t) = X(t)^2 Y(t)^2$$

is WSS. Find the power generated by $Z_3(t)$. (Note: This is the only part of the problem where you use the fact that $X(t)$ and $Y(t)$ are both Gaussian processes.)

Supplementary Problems: (not to hand in) From the textbook, you can try Problems 10.6.1, 10.8.3, 10.9.2, 10.10.1, 10.12.1