EE 3025 S2005 Homework Set #11 Solutions

Mr. AlHussien is grading Problem 5

Mr. Msechu is grading Problems 1, 2(a)(b)

Solution to Problem 1:

Solution to (a):

```
p=1/3;
x=floor(log(rand(1,100000))/log(p));
for i=1:10
p_estimated(i)=mean(x==i-1);
end
k=0:9;
p_actual=(p.^k).*(1-p);
p_actual

p_actual =

  Columns 1 through 7

    0.6667    0.2222    0.0741    0.0247    0.0082    0.0027    0.0009

  Columns 8 through 10

    0.0003    0.0001    0.0000

p_estimated

p_estimated =

  Columns 1 through 7

    0.6657    0.2235    0.0733    0.0248    0.0087    0.0027    0.0009

  Columns 8 through 10

    0.0003    0.0001    0.0001
```

In every case, there is agreement to two decimal places.

```
p=2/3;
x=floor(log(rand(1,100000))/log(p));
for i=1:10
p_estimated(i)=mean(x==i-1);
end
k=0:9;
```

```
p_actual=(p.^k).*(1-p);
p_actual

p_actual =

  Columns 1 through 7

    0.3333    0.2222    0.1481    0.0988    0.0658    0.0439    0.0293

  Columns 8 through 10

    0.0195    0.0130    0.0087

p_estimated

p_estimated =

  Columns 1 through 7

    0.3363    0.2213    0.1458    0.0973    0.0665    0.0448    0.0298

  Columns 8 through 10

    0.0189    0.0132    0.0086
```

In every case except the first, you get agreement to two decimal places.

Solution to (b):

```
p=.1:.1:.9;
for j=1:9
S=0;
v=floor(log(rand(1,50001))/log(p(j)));
for i=2:50001
q(i)=max(q(i-1)+v(i)-1,v(i));
S=S+q(i);
end
q_average(j)=S/50000;
end
format long
q_average =

   1.0e+05 *

  Columns 1 through 4

   0.0000012404    0.0000033660    0.0000074978    0.0000196012
```

```
Columns 5 through 8

 0.0017771142   0.1263883438   0.3289536212   0.7554165878

Column 9

 2.0080963080
```

p	average of Q_n
.1	0.1240
.2	0.3366
.3	0.7498
.4	1.9601
.5	177.7114
.6	12638.8344
.7	32895.3621
.8	75541.6588
.9	200809.6308

Solution to (c): It looks like stability occurs for $p < .5$ and instability occurs for $p \geq .5$. Here is an intuitive reason why this happens. If you look at the single-server queue model in Recitation 13 instructions, you see that the queueing system is stable if and only if $\mu > \lambda$, where μ is the service rate and λ is the arrival rate. Let us suppose that the condition $\mu > \lambda$ is also the necessary and sufficient condition for stability of the queue in Problem 1. In Problem 1, we have

$$\begin{aligned} \mu &= 1 \\ \lambda &= \frac{1}{1-p} - 1 \end{aligned}$$

The inequality $\mu > \lambda$ becomes the inequality

$$1 > \frac{1}{1-p} - 1.$$

Solving this inequality for p, you get

$$p < 1/2$$

as the necessary and sufficient condition for stability.

Solution to Problem 2:

Solution to (a): In class notes, we showed

$$R_{Z_1}(\tau) = R_X(\tau)R_Y(\tau).$$

Therefore,

$$R_{Z_1}(\tau) = (16 + 5\exp(-|\tau|))(25 + 20\exp(-|\tau|)).$$

Plugging in $\tau = 0$, we see that

$$P_{Z_1} = R_{Z_1}(0) = R_X(0)R_Y(0) = 21 * 45 = 945.$$

Solutiuon to (b): In class notes, we showed

$$R_{Z_2}(\tau) = R_X(\tau) + R_Y(\tau) + 2\mu_X\mu_Y.$$

Therefore,

$$R_{Z_2}(\tau) = 1 + 25\exp(-|\tau|).$$
$$P_{Z_2} = R_{Z_2}(0) = 26.$$

Solution to (c): It turns out that the Z_3 process is WSS. Therefore,

$$P_{Z_3} = E[Z_3(t)^2] = E[X(t)^4 Y(t)^4] = E[X(t)^4]E[Y(t)^4].$$

The rest of the problem requires only Chapter 2-3 skill level. If Z is Gaussian(0,1), we showed in class that $E[Z^4] = 3$. If S is Gaussian with mean μ and std dev σ, we can model S as

$$S = \sigma Z + \mu.$$

Therefore,

$$\begin{aligned} E[S^4] &= E[(\sigma Z + \mu)^4] \\ &= \sigma^4 E[Z^4] + 4\sigma^3\mu E[Z^3] + 6\sigma^2\mu^2 E[Z^2] + 4\sigma\mu^3 E[Z] + \mu^4 \\ &= 3\sigma^4 + 6\sigma^2\mu^2 + \mu^4 \end{aligned}$$

The Gaussian RV $X(t)$ has mean $\mu = -4$ and variance $\sigma^2 = 5$. Therefore

$$E[X(t)^4] = 3\sigma^4 + 6\sigma^2\mu^2 + \mu^4 = 811.$$

The Gaussian RV $Y(t)$ has mean $\mu = 5$ and variance $\sigma^2 = 20$. Therefore

$$E[Y(t)^4] = 3\sigma^4 + 6\sigma^2\mu^2 + \mu^4 = 4825.$$

We conclude that

$$P_{Z_3} = 811 * 4825 = 3913075.$$

In my office hours, I noticed that several students tried to compute $R_{Z_3}(\tau)$. It is possible to do this, difficult to do this, and not necessary to do this. None of these students did it correctly. You have to compute

$$R_{Z_3}(\tau) = E[X(t)^2 X(t+\tau)^2]E[Y(t)^2 Y(t+\tau)^2].$$

For example, to compute $E[X(t)^2 X(t+\tau)^2]$, you can find constants a, b, c, d so that if Z_1, Z_1 are independent Gaussian(0,1) RV's, then $X(t)$ and $X(t+\tau)$ can be modeled as

$$\begin{aligned} X(t) &= aZ_1 + bZ_2 \\ X(t+\tau) &= cZ_1 + dZ_2 \end{aligned}$$

Then you can compute

$$E[X(t)^2 X(t+\tau)^2] = E[(aZ_1 + bZ_2)^2 (cZ_1 + dZ_2)^2]$$

by expanding the right side as a sum of several terms which are each of form a constant times $E[Z_1^i]E[Z_2^j]$ for various powers i, j. It's tedious but doable.

Solution to Problem 3:

Solution to (a): Let the random number drawn on Step 1 be denoted Q. If Step 1 results in $Q = q$, then by the law of large numbers, the time averages of the X realizations on Step 2 converge to $2q - 1$, which is the expected value of each X_n sample given $Q = q$. Therefore,

$$\lim_{N\to\infty} \left(\frac{X_1 + X_2 + \cdots + X_N}{N} \right) = 2Q - 1, \tag{1}$$

where the limit is in the stochastic convergence sense. Since the limit is random, the X process cannot be ergodic.

Solution to (b): Use law of iterated expectation as we did in the similar Problem 2 on Homework Set 9:

$$\mu_X = E[X_n] = E[E[X_n|Q]] = E[2Q-1] = 2E[Q] - 1 = 2(1/2) - 1 = 0.$$

$$P_X = E[X_n^2] = 1,$$

because X_n^2 is always equal to 1.

$$\sigma_X^2 = P_X - \mu_X^2 = 1.$$

If $\tau \neq 0$,

$$\begin{aligned} R_X(\tau) &= E[X_n X_{n+\tau}] \\ &= E[E[X_n X_{n+\tau}|Q]] \\ &= E[E[X_n|Q]E[X_{n+\tau}|Q]] \\ &= E[(2Q-1)^2] = 4E[(Q-0.5)^2] = 4\sigma_Q^2 = 4(1/12) = 1/3 \end{aligned}$$

We conclude that

$$R_X(\tau) = \begin{cases} 1, & \tau = 0 \\ 1/3, & \tau \neq 0 \end{cases}$$

Solution to (c): Notice that

$$\lim_{\tau\to\infty} R_X(\tau) = 1/3 \neq \mu_X^2.$$

Therefore the process cannot be ergodic.

Solution to (d): From (1), we see that for large N, the average

$$\bar{X}_N = (X_1 + X_2 + X_3 + \cdots + X_N)/N$$

is a good estimate of $2Q - 1$. Adding one and dividing by two, we conclude that

$$\frac{\bar{X}_N + 1}{2} = (0.5)\bar{X}_N + (0.5) \quad (2)$$

must be a good estimator of Q. If you are interested, you can push this even further. You can find the "slope parameter" a_N and the "intercept parameter" b_N such that

$$a_N \bar{X}_N + b_N$$

is the minimum mean square straight line estimator of Q based on $\bar{X}_N$ (similar to what was done in Problem 2 of Homework 9 for $N = 3$). In view of (2), my guess is that a_N and b_N will both converge to 1/2. You can also compute the MS estimation error

$$E[(Q - \{a_N \bar{X}_N + b_N\})^2]$$

and show that it goes to zero as $N \to \infty$. These remaining things that I've suggested to do fall within the purview of an area of statistics called *Bayesian statistics*. In Bayesian statistics, a nonergodic process can be a good thing because you may be able to estimate a randomly selected parameter by looking at time averages along the process.

Solution to Problem 4:

Solution to (a): The input power is

$$\text{input power } = \int_{-B}^{B} df = 2B.$$

The output power spectrum is

$$S_Y(f) = |H(f)|^2 = \frac{1}{1 + (RC2\pi f)^2},$$

for $|f| \leq B$, and equal to zero elsewhere. Therefore, the output power is

$$P_Y = \int_{\infty}^{\infty} S_Y(f) df = \int_{-B}^{B} \frac{1}{1 + (RC2\pi f)^2} df = (1/\pi RC)\text{Tan}^{-1}(RC2\pi B).$$

Setting output power 75% of input power, we obtain the equation

$$RC2\pi B = (4/3) * \text{Tan}^{-1}(RC2\pi B),$$

which we cannot explicitly solve for RC in terms of B because the equation is transcendental in RC. My approach was first to solve the transcendental equation

$$x = (4/3)Tan^{-1}(x) \quad (3)$$

for some positive solution x. By the "iteration method", Matlab readily found such an x:

```
x=1;
for i=1:1000
x=(4/3)*atan(x);
end
x

x =

   1.12630779127771

4/3*atan(x)

ans =

   1.12630779127771
```

It appears that one possible solution to (3) is

$$x = 1.12630779127771.$$

Setting this equal to $2\pi RCB$, we obtain the following relationship between B and RC:

$$RC = \frac{1.12630779127771}{2\pi B}.$$

Plugging in $B = 2000\pi$, you get

$$RC = \frac{1.12630779127771}{4000\pi^2} = 0.00002892971.$$

Solution to (b): The transcendental relationship between RC and B is now

$$RC2\pi B = (8/7) * \mathrm{Tan}^{-1}(RC2\pi B),$$

so we now solve

$$x = (8/7)Tan^{-1}(x)$$

for x:

```
x=1;
for i=1:1000
x=(8/7)*atan(x);
end
x
x =

   0.69117718193292
```

Therefore,

$$RC2\pi B = 0.69117718193292,$$

and so

$$B = \frac{0.69117718193292}{RC2\pi} = \frac{69.117718193292}{2\pi} = 11.$$

Solution to Problem 5:

Solution to (a): If the input is $\delta(t)$, then the output is $h(t)$. So obviously

$$h(t) = 4\delta(t) - 3\delta(t-1) + 2\delta(t-4).$$

Fourier transforming $h(t)$, we get the frequency response:

$$H(f) = 4 - 3\exp(-j2\pi f) + 2\exp(-j8\pi f).$$

Solution to (b): Use the formula

$$\delta(t-t_0) * \delta(t-t_1) = \delta(t-[t_0+t_1]).$$

Then you easily convolute

$$h(-t) = 4\delta(t) - 3\delta(t+1) + 2\delta(t+4)$$

with $h(t)$. You get

$$h(t)*h(-t) = 29\delta(t)+8[\delta(t+4)+\delta(t-4)]-6[\delta(t+3)+\delta(t-3)]-12[\delta(t+1)+\delta(t-1)]. \tag{4}$$

You now convolute this result with $R_X(\tau)$, exploiting the relationship

$$R_X(\tau) * \delta(\tau-\tau_0) = R_X(\tau-\tau_0),$$

You get:

$$R_Y(\tau) = 87e^{-.5|\tau|}+24[e^{-.5|\tau+4|}+e^{-.5|\tau-4|}]-18[e^{-.5|\tau+3|}+e^{-.5|\tau-3|}]-36[e^{-.5|\tau+1|}+e^{-.5|\tau-1|}].$$

(c) Look at equation (4). Transform each term, using the fact that

$$\mathcal{F}[\delta(t+t_0)+\delta(t-t_0)] = 2\cos(2\pi t_0 f).$$

This gives us

$$|H(f)|^2 = 29 + 16\cos(8\pi f) - 12\cos(6\pi f) - 24\cos(2\pi f).$$

We also have

$$S_X(f) = \frac{3}{(2\pi f)^2 + 0.25}.$$

It follows that

$$S_Y(f) = |H(f)|^2 S_X(f) = 3\left(\frac{29 + 16\cos(8\pi f) - 12\cos(6\pi f) - 24\cos(2\pi f)}{(2\pi f)^2 + 0.25}\right).$$

(d) One way to find P_Y is

$$P_Y = R_Y(0) = 87 + 48\exp(-2) - 36\exp(-1.5) - 72\exp(-.5) = 41.7932.$$

The other way is

$$P_Y = \int_{-\infty}^{\infty} S_Y(f)df = 3\int_{-\infty}^{\infty} \frac{29 + 16\cos(8\pi f) - 12\cos(6\pi f) - 24\cos(2\pi f)}{(2\pi f)^2 + 0.25}df.$$

Here is a Matlab script evaluating this integral:

```
syms f
I=int((29+16*cos(8*pi*f)-12*cos(6*pi*f)-24*cos(2*pi*f))/((2*pi*f)^2+.25),f,-inf,inf);
power=eval(3*I)
power =

   41.7932
```