EE 3025 S2005 Homework Set #12 Solutions

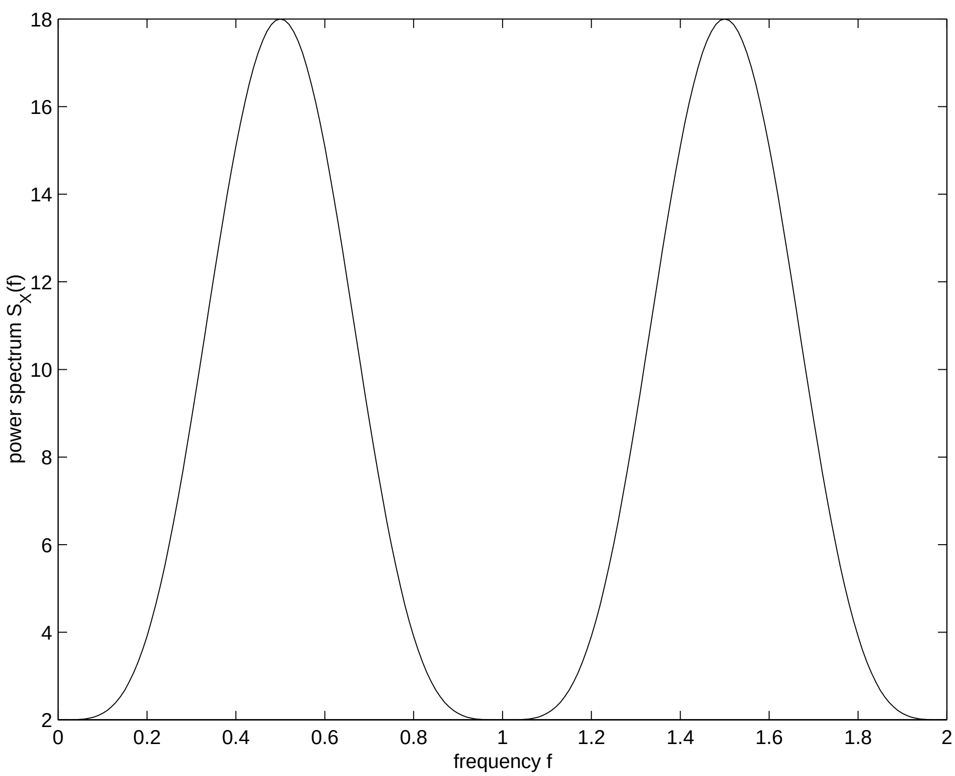
(We will be grading Problems 1-3)

Solution to Problem 1:

Solution to (a): Fourier transforming $R_X(\tau)$, you get

$$\begin{aligned} S_X(f) &= 8 - 4\exp(-j2\pi f) - 4\exp(j2\pi f) + \exp(-4\pi f) + \exp(4\pi f) \\ &= 8 - 8\cos(2\pi f) + 2\cos(4\pi f) \end{aligned}$$

```
f=0:.01:2;
SXf=8-8*cos(2*pi*f)+2*cos(4*pi*f);
plot(f,SXf)
```

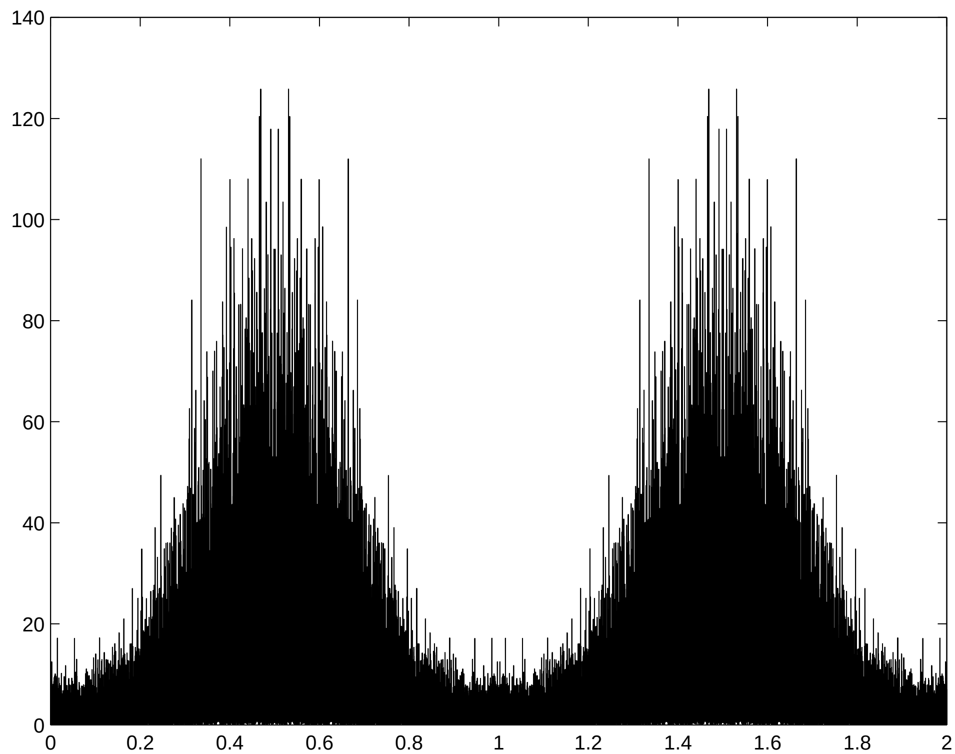


Solution to (b):

```
N = ; %enter here the number of samples N you want
a=1+sqrt(2);b=-sqrt(2);c=-1+sqrt(2);
z=randn(1,N+2);
x=a*z(3:N+2)+b*z(2:N+1)+c*z(1:N);
```

Solution to (c): The fft works best with a number of samples equal to a power of two. I decided to use 32768 samples.

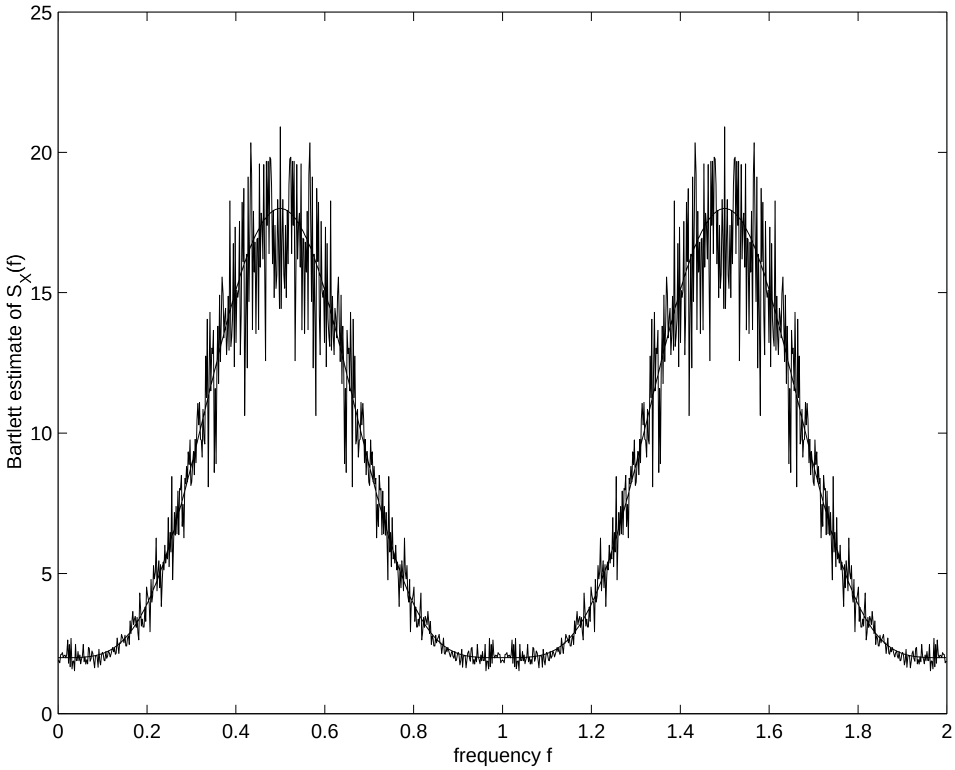
```
N=32768;
a=1+sqrt(2);b=-sqrt(2);c=-1+sqrt(2);
z=randn(1,N+2);
x=a*z(3:N+2)+b*z(2:N+1)+c*z(1:N);
pgram = abs(fft(x)).^2/N;
f=(0:2*N-1)/N;
plot(f,[pgram pgram])
```



Solution to (d): I decided to use 32768 samples in total, divided up into 64 groups of 512 samples each.

```
clear
N1=512;
N2=64;
N=N1*N2;
a=1+sqrt(2);b=-sqrt(2);c=-1+sqrt(2);
z=randn(1,N+2);
x=a*z(3:N+2)+b*z(2:N+1)+c*z(1:N);
s=zeros(1,N1);
for j=1:N2
segment=x((j-1)*N1+1:j*N1);
periodogram=abs(fft(segment)).^2/N1;
```

```
s=s+periodogram;
end
SXhat=s/N2;
f=(0:2*N1-1)/N1;
plot(f,[SXhat SXhat])
```



Solution to Problem 2:

Solution to (a): Looking at the earlier problem's solution, you see that the periodic $R_X(\tau)$ is equal to $1-4\tau$ for $0 \leq \tau \leq 1/2$. This is all you need. The k-th Fourier coefficient of $R_X(\tau)$ is then computable as

$$\begin{aligned} a_k &= \int_{-1/2}^{1/2} R_X(\tau)\exp(-jk2\pi\tau)d\tau \\ &= 2\int_0^{1/2}(1-4\tau)\cos(jk2\pi\tau)d\tau \end{aligned}$$

You can integrate by parts or use Matlab. You get

$$a_k = \begin{cases} 0, & k = 0, \pm 2, \pm 4, \pm 6, \cdots \\ \frac{4}{k^2\pi^2}, & k = \pm 1, \pm 3, \pm 5, \cdots \end{cases}$$

We can write the power spectrum as

$$S_X(f) = \sum_{j=1}^{\infty}\left(\frac{4}{(2j-1)^2\pi^2}\right)[\delta(f-2j+1)+\delta(f+2j-1)].$$

Solution to (b): For bandwidth $B = 1.5$ or bandwidth $B = 2.5$, the filter output power is $8/\pi^2$, which is 81.06% of the input power (since the input power is 1). For bandwidth $B = 3.5$ or bandwidth $B = 4.5$, the filter output power is $(8/\pi^2) + 8/(9\pi^2)$, which is 90.06% of the input power. For bandwidth $B = 5.5$, the filter output power is $(4/\pi^2) + 4/(9\pi^2) + 4/(25\pi^2)$, which is 93.31% of the input power.

bandwidth	power ratio percentage
1.5	81.06
2.5	81.06
3.5	90.06
4.5	90.06
5.5	93.31

Solution to (c): $B = 3.5$ is the smallest of the bandwidths that you were given to choose from, for which the power ratio is at least 90%. This is clear from the table constructed in the solution to (b).

Solution to Problem 3:

Solution to (a): Referring to Section 40.4 of the class notes, you see that the system to solve is

$$\begin{bmatrix} R_Y(0) & R_Y(1) & R_Y(2) \\ R_Y(1) & R_Y(0) & R_Y(1) \\ R_Y(2) & R_Y(1) & R_Y(0) \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \\ h[2] \end{bmatrix} = \begin{bmatrix} R_X(0) \\ R_X(1) \\ R_X(2) \end{bmatrix},$$

which reduces in this case to

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \\ h[2] \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix},$$

because

$$\begin{aligned} R_X(0) &= 2 \\ R_X(1) &= 1 \\ R_X(2) &= 0 \\ R_Y(0) &= R_X(0) + R_Z(0) = 3 \\ R_Y(1) &= R_X(1) + R_Z(1) = 1 \\ R_Y(2) &= R_X(2) + R_Z(2) = 0 \end{aligned}$$

The solutions are

$$h[0] = 13/21, \quad h[1] = 1/7, \quad h[2] = -1/21.$$

Solution to (b): We have

$$\begin{aligned} E[X_n Y_n] &= E[X_n^2] + E[X_n Z_n] = R_X(0) + 0 = 2 \\ E[X_n Y_{n-1}] &= E[X_n X_{n-1}] + E[X_n Z_{n-1}] = R_X(1) + 0 = 1 \\ E[X_n Y_{n-2}] &= E[X_n X_{n-2}] + E[X_n Z_{n-2}] = R_X(2) + 0 = 0 \end{aligned}$$

The MS estimation error is then

$$R_X(0) - h[0](2) - h[1](1) - h[2](0) = 13/21.$$

In decibels, this is

$$10\log_{10}\frac{2}{13/21} = 5.09\ \textit{decibels}.$$

Solution to (c): We have

$$\begin{aligned} S_X(f) &= 2 + 2\cos(2\pi f) \\ S_Z(f) &= 1 \end{aligned}$$

This gives us

$$E_{Wiener} = \int_0^1 \frac{2 + 2\cos(2\pi f)}{3 + 2\cos(2\pi f)} df = 1 - \sqrt{5}/5 = 0.5528.$$

In decibels, this is

$$10\log_{10}\frac{2}{.5528} = 5.58\ \textit{decibels}.$$

Our conclusion is that you can improve about half a decibel in system performance if you use a more sophisticated receiver than the one in part(a).

Solution to Problem 4:

Solution to (a): The decibel figure without filtering is

$$10\log_{10}\frac{P_X}{P_Z} = 10\log_{10}(2) = 3.01\ \textit{decibels}.$$

Solution to (b): The signal part of the output power is the matrix triple product

$$\begin{bmatrix} h[0] & h[1] & h[2] \end{bmatrix} \begin{bmatrix} R_X(0) & R_X(1) & R_X(2) \\ R_X(1) & R_X(0) & R_X(1) \\ R_X(2) & R_X(1) & R_X(0) \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \\ h[2] \end{bmatrix}.$$

For the filter coefficients in (a) of Problem 4, this gives us

$$\begin{bmatrix} 13/21 & 1/7 & -1/21 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 13/21 \\ 1/7 \\ -1/21 \end{bmatrix} = 0.9751.$$

The noise part of the output power is

$$h[0]^2 + h[1]^2 + h[2]^2 = (13/21)^2 + (1/7)^2 + (-1/21)^2 = 0.4069.$$

The SNR is therefore

$$10\log_{10}\frac{0.9751}{0.4069} = 3.81\ \textit{decibels}.$$

Solution to (c): Simply adapt the Matlab script provided in Example 41.1 of the class notes:

```
R=toeplitz([2 1 0]);
[a,b]=eig(R)
a =

    0.5000   -0.7071   -0.5000
    0.7071    0.0000    0.7071
    0.5000    0.7071   -0.5000


b =

    3.4142         0         0
         0    2.0000         0
         0         0    0.5858
```

The largest eigenvalue of the 3×3 correlation matrix of the X samples is clearly 3.4142. The corresponding eigenvector is therefore the first column of the matrix a. We can base our tap weights on this eigenvector:

$$h[0] = 0.5000, \quad h[1] = 0.7071, \quad h[2] = 0.5000.$$

We indeed have

$$h[0]^2 + h[1]^2 + h[2]^2 = 1$$

and so the resulting SNR (which is maximal) is

$$10 \log_{10} \frac{3.4142}{1} = 5.33 \textit{ decibels}.$$

(The power due to the signal part of the receiver output will always be the largest eigenvalue of the 3×3 correlation matrix of the X samples, for the max SNR filter.)

Solution to Problem 5:

Solution to (a): The mean function of the output process $X(t)$ is zero because the mean function of white noise is zero and we are doing linear filtering. Therefore,

$$E[X(4)] = 0.$$

By the double integral trick explained in the Lecture 41 notes, we have

$$Var(X(4)) = E[X(4)^2] = \int_0^4 \int_0^4 s_1 s_2 \delta(s_1 - s_2) ds_1 ds_2.$$

We have to compute

$$\int_0^4 s_1 \delta(s_1 - s_2) ds_1.$$

By the "sifting property" of the delta function,

$$s_1\delta(s_1 - s_2) = s_2\delta(s_1 - s_2),$$

where here s_1 is the variable and s_2 is held fixed. Also we have the following indefinite integral:

$$\int \delta(s_1 - s_2)ds_1 = u(s_1 - s_2).$$

It follows that

$$\int_0^4 s_1\delta(s_1 - s_2)ds_1 = s_2[u(4 - s_2) - u(-s_2)].$$

It is easy to see that $u(4 - s_2) - u(-s_2)$, as a function of s_2, is a rectangular pulse of amplitude 1 that goes from $s_2 = 0$ to $s_2 = 4$. We conclude that

$$\begin{aligned} Var[X(4)] &= \int_0^4 s_2 \left[\int_0^4 s_1\delta(s_1 - s_2)ds_1\right] ds_2 \\ &= \int_0^4 s_2^2[u(4 - s_2) - u(-s_2)]ds_2 \\ &= \int_0^4 s_2^2 ds_2 = 64/3. \end{aligned}$$

Let $f(x_4)$ denote the density of $X(4)$. Since $X(4)$ is Gaussian with mean 0, we must have

$$f(x_4) = \frac{1}{\sqrt{2\pi}\sigma_{X(4)}} \exp(-x_4^2/2\sigma_{X(4)}^2) = \frac{1}{\sqrt{128\pi/3}} \exp(-3x_4^2/128).$$

Solution to (b): Using independent increments property,

$$\begin{aligned} Cov(X(4), X(7)) &= E[X(4)X(7)] \\ &= E[X(4)(X(7) - X(4))] + E[X(4)]^2 \\ &= E[X(4)]E[X(7) - X(4)] + 64/3 = 0 + 64/3 = 64/3 \end{aligned}$$

The correlation coefficient ρ is given by

$$\rho = \frac{Cov(X(4), X(7))}{\sqrt{Var(X(4))Var(X(7))}}.$$

Similarly to what was done in (a), one can show that

$$Var(X(7)) = 7^3/3 = 343/3.$$

Therefore,

$$\rho = \frac{64/3}{\sqrt{(64/3)(343/3)}} = \frac{8}{\sqrt{343}} = 0.4320.$$

Let $f(x_4, x_7)$ denote the joint PDF of $(X(4), X(7))$. Plugging into the form of the bivariate Gaussian density on page 191 of your textbook, you get

$$f(x_4, x_7) = \frac{3}{16\pi\sqrt{279}} \exp\left[-\frac{1029(x_4^2/64) - 6x_4x_7 + 3x_7^2}{558}\right].$$