EE 3025 S2005 Homework Set #1 Solutions
We will grade Problems 1,2,4

Solution to Problem 1. The sample space consists of the 216 triples (i,j,k) in which i,j,k are each 1 through 6. The probability of an event is the number of triples (i,j,k) in E divided by 216.

$$P(A) = P(111, 222, 333, 444, 555, 666) = 6/216 = 1/36.$$

$$P(B) = P(113, 131, 311, 122, 212, 221) = 6/216 = 1/36.$$

$$P(C) = 3^3/216 = 1/8,$$

because each entry is either 1,3,5.

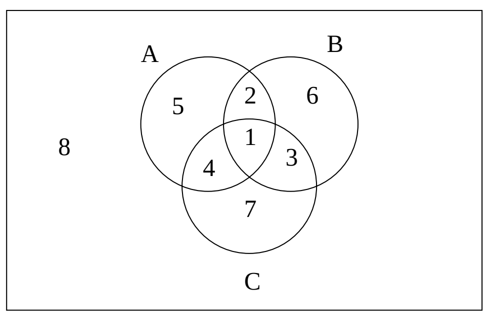
$$P(A \cap B) = P(empty\ set) = 0.$$

$$P(A \cap C) = P(111, 333, 555) = 3/216 = 1/72.$$

$$P(B \cap C) = P(113, 131, 311) = 3/216 = 1/72.$$

$$P(A \cap B \cap C) = 0.$$

Solution to Problem 2.



Looking at the Venn Diagram, we see that

$$P(1) = 0.01$$

$$P(2) = P(A \cap B) - P(1) = 0.13 - 0.01 = 0.12.$$

$$P(4) = P(A \cap C) - P(1) = 0.07 - 0.01 = 0.06$$

$$P(A \cap B^c \cap C^c) = P(5) = P(A) - P(1) - P(2) - P(4) = 0.38.$$

$$\begin{aligned} P(A \cup B^c) &= P(A) + P(B^c) - P(A \cap B^c) \\ &= .57 + (1 - .49) - P(4) - P(5) \\ &= .57 + (1 - .49) - 0.06 - 0.38 = 0.64. \end{aligned}$$

Solution to Problem 3. Using the same Venn Diagram and the hint in Example 3.9 of the class notes, you obtain the following system of equations to solve:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P(1) \\ P(2) \\ P(4) \\ P(3) \\ P(6) \\ P(5) \\ P(7) \\ P(8) \end{bmatrix} = \begin{bmatrix} 1 \\ 0.25 \\ 0.2 \\ 0.25 \\ 0.1 \\ 0.05 \\ 0 \\ 0.3 \end{bmatrix}$$

The solution is unique and is

$$\begin{aligned} P(1) &= 0.05 \\ P(2) &= 0.05 \\ P(4) &= 0.15 \\ P(3) &= 0.05 \\ P(6) &= 0.05 \\ P(5) &= 0 \\ P(7) &= 0 \\ P(8) &= 0.65 \end{aligned}$$

$$P(A^c \cap B^c \cap C^c) = P(8) = 0.65.$$

$$\begin{aligned} P(fails\ exactly\ one\ test) &= P(A \cap B^c \cap C^c) + P(A^c \cap B \cap C^c) + P(A^c \cap B^c \cap C) \\ &= P(5) + P(6) + P(7) = 0.05 \end{aligned}$$

Solution to Problem 4. Examine Problem 7.2 of the Chapter 1 Solved Problems. Denoting the teams as 1 and 2, the sample space S is the set of all words formed from between 4 and 7 entries from the set $\{1, 2\}$, in which

- If the word ends in 1, then there are exactly three previous 1's and three or fewer previous 2's.

- If the word ends in 2, then there are exactly three previous 2's and three or fewer previous 1's.

(If you were to take the trouble to develop the very large tree for this experiment, you'd get the words described above in following the root-to-leaf paths.) From the tree model, each word in the sample space has probability equal to $1/2^L$, where L is the length of the word.

part(a). We want $P(E_1)$, where E_1 is the set of all words in S beginning and ending with different entries. We can catalog the words in E_1 (and their probabilities) as follows:

- There are no words in E_1 of length 4.

- There are two words in E_1 of length 5: 12222,21111. They each have prob 1/32.
- There are eight words in E_1 of length 6:

$$112222, 121222, 122122, 122212$$

$$221111, 212111, 211211, 211121$$

  They each have prob 1/64.
- There are $2\binom{5}{2} = 20$ words in E_1 of length 7. You can list them if you want. (Choose the middle 5 entries to consist of 2 1's and 3 2's or vice-versa.) They each have probability 1/128.

We conclude that

$$P(E_1) = 2(1/32) + 8(1/64) + 20(1/128) = 0.3438.$$

part(b). We want $P(E_2)$, where E_2 is the set of all words in S beginning with 11 and ending in 2 or beginning in 22 and ending in 1. We can catalog the words in E_2 (and their probabilities) as follows:

- There are no words in E_2 of length 4 or 5.
- There are two words in E_2 of length 6: 112222,221111. They each have prob 1/64.
- There are eight words in E_1 of length 7:

$$1112222, 1121222, 1122122, 1122212$$

$$2221111, 2212111, 2211211, 2211121$$

  They each have prob 1/128.

We conclude that

$$P(E_2) = 2(1/64) + 8(1/128) = 0.0938.$$

part(c). We want $P(E_3)$, where E_3 is the set of all words in S beginning with something from $\{112, 121, 211\}$ and ending in 2 or beginning with something from $\{221, 212, 122\}$ and ending with 1. We can catalog the words in E_3 (and their probabilities) as follows:

- There are no words in E_3 of length 4 or 5.
- There are 6 words in E_3 of length 6:

$$112222, 121222, 211222, 221111, 212111, 122111$$

  They each have prob 1/64.
- There are 18 words in E_3 of length 7. To see this, start with a word from $\{112, 121, 211\}$, choose two of next three entries to be 2, and end with 2. This gives $3 * 3 = 9$ members of E_3. The other 9 are obtained by switching the 1's and 2's around.

We conclude that

$$P(E_3) = 6(1/64) + 18(1/128) = 0.2344.$$

Solution to Problem 5.
part(a). The answer is

$$P(H, TTTH, TTTTTTH, TTTTTTTTTH, \cdots).$$

This is the infinite series

$$1/2 + (1/2)^4 + (1/2)^7 + (1/2)^{10} + \cdots$$

which sums up to

$$\frac{1/2}{1-(1/2)^3} = 4/7.$$

part(b). The answer is

$$P(TH, TTTTH, TTTTTTTH, \cdots).$$

Every outcome has prob 1/2 of one of Bill's winning outcomes in (a). So the answer is

$$2/7.$$

part(c). Similarly, the answer has to be one half the answer to part(b), or

$$1/7.$$

Or, since the (a),(b),(c) answers must add up to 1, the answer to (c) is

$$1 - (4/7) - (2/7) = 1/7.$$