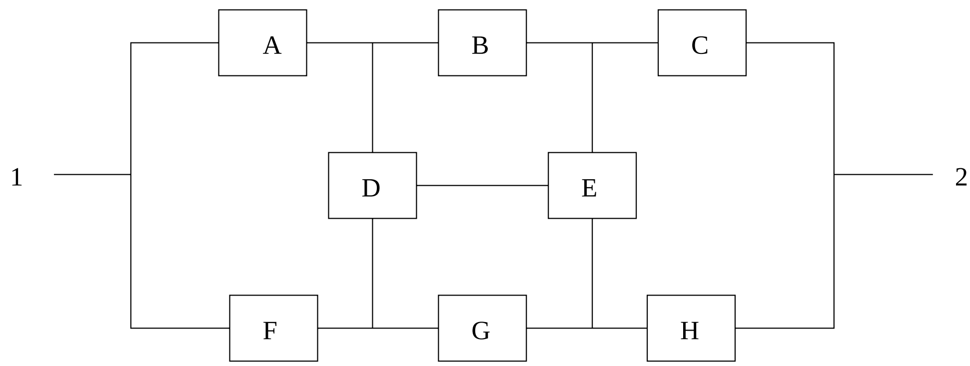
EE 3025 S2005 Homework Set #3 Solutions

Problem 1 to be graded by Mr. AlHussien
Problems 2,5 to be graded by Mr. Msechu
(each graded problem worth 10 POINTS)

1. Consider the relay circuit in the following diagram:



Relay switches A, B, C, D, E, F, G, H operate independently; each of them has the same probability p of working correctly. Let $P(1 \rightarrow 2)$ denote the probability that a connection will be made from point 1 to point 2. Obviously, $P(1 \rightarrow 2)$ will be a complicated function of p. In this problem, you are not going to find the explicit formula for $P(1 \rightarrow 2)$ as a function of p. Instead, you will be using Matlab simulations of the circuit to obtain estimates of $P(1 \rightarrow 2)$ for various p values. Before starting on this Matlab problem, you might want to study Experiment 3 of Recitation 2.

(a) Use Matlab to do 50000 simulated trials of the circuit with $p = 0.90$. Estimate $P(1 \rightarrow 2)$ based on these trials. You should run your 50000 trials more than once to see if your estimate is "robust" (that is, hopefully your estimate will not change very much).

Solution. The paths I used are ABC, ABEH, ADEH, ADGH, ADEC, FGH, FGEC, FDBC, FDEC, FDEH. (There are also paths ADGEC and FDBEH, but if these provide a connection, so do the paths ADEC,FDEH already listed.)

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p=0.90;
M=(rand(8,50000)<p);
%xA=M(1,:);xB=M(2,:);xC=M(3,:);xD=M(4,:);
%xE=M(5,:);xF=M(6,:);xG=M(7,:);xH=M(8,:);
xABC=min(M([1 2 3],:));
xABEH=min(M([1 2 5 8],:));
xADEH=min(M([1 4 5 8],:));
xADGH=min(M([1 4 7 8],:));
xADEC=min(M([1 4 5 3],:));
xFGH=min(M([6 7 8],:));
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xFGEC=min(M([6 7 5 3],:));
xFDBC=min(M([6 4 2 3],:));
xFDEC=min(M([6 4 5 3],:));
xFDEH=min(M([6 4 5 8],:));
x12=max([xABC;xABEH;xADEH;xADGH;xADEC;xFGH;xFGEC;xFDBC;xFDEC;xFDEH]);
mean(x12)
```

The estimates seemed to fluctuate about 0.975.

(b) By trial and error, do 50000 simulated trials over and over again with different values of p on each run of 50000, until you find a value of p for which the estimate for $P(1 \to 2)$ is as close to 0.85 as you can get.

Solution. Around $p = 0.77$ seems right.

2. This problem, as part of its solution, will involve the binomial distribution in some way.

(a) Flip 10 fair dies. Compute the probability that at least 7 of the dies yield a number ≥ 5.

Solution.

$$Answer(setup) = \sum_{k=7}^{10} \binom{10}{k} (1/3)^k (2/3)^{10-k}$$

(b) Select 10 light bulbs and let each of them burn until they burn out. Each light bulb has a lifetime which is exponentially distributed with mean lifetime 1000 hours. Compute the probability that between 3 and 6 of the light bulbs (inclusively) burn for at least 1200 hours.

Solution. You have a Binomial(n,p) distribution where $n = 10$ and

$$p = \int_{1200}^{\infty} a \exp(-xa) dx, \quad a = .001.$$

$$p = 0.3012.$$

$$Answer(setup) = \sum_{k=3}^{6} \binom{10}{k} (0.3012)^k (0.6988)^{10-k}.$$

3. Dropout is a kind of defect in magnetic tape in which the signal disappears for a brief period of time. We assume that the number of dropouts in a fixed length of tape is a Poisson random variable.

(a) Suppose the number of dropouts in an 1800ft length of tape is a Poisson RV with parameter 3. What is the probability of 3 or more dropouts in this length of tape? What is the conditional probability of 3 or more dropouts given that there is at least one dropout?

Solution.

$$P(3 \text{ or more}) = 1 - \sum_{k=0}^{2} \exp(-3) 3^k / k!$$

$$\begin{aligned} P(3 \text{ or more} | 1 \text{ or more}) &= P(3 \text{ or more}) / P(1 \text{ or more}) \\ &= (\text{previous answer}) / (1 - \exp(-3)) \end{aligned}$$

(b) Suppose again that the number of dropouts in an 1800ft length of tape is a Poisson RV with parameter 3. Let random variable X be the number of dropouts in a 3600ft of tape. X is a Poisson random variable. What is the parameter? Compute the probability that $X \geq 6$.

Solution. The parameter is 6, twice as much as before.

$$Answer(setup) = P(6 \text{ or more}) = 1 - \sum_{k=0}^{5} \exp(-6)6^k/k!.$$

4. A mixed random variable X has probability density function

$$f_X(x) = (1/6)\delta(x-5) + (1/6)\delta(x-10) + (1/6)\delta(x-15) + \frac{1}{20\sqrt{2\pi}} \exp\left(-\frac{(x-20)^2}{200}\right),$$

defined for all real x.

(a) Attach a Matlab plot of the cumulative distribution function $F_X(x)$ over the range $0 \leq x \leq 40$.

Solution. The integrator output has three unit step function components and a scaled Gaussian(μ, σ) CDF component with $\mu = 20$ and $\sigma = 10$. Specifically,

$$F_X(x) = (1/6)u(x-5) + (1/6)u(x-10) + (1/6)u(x-15) + (1/2)\Phi((x-20)/10).$$

Now use the fact that

$$\Phi(z) = (1/2) + (1/2)\text{erf}(z/\sqrt{2}).$$

```
x=0:0.01:40;
y1=(x>=5); y2=(x>=10); y3=(x>=15);
z=(x-20)/10; y4=1/2 + 1/2*erf(z/sqrt(2));
y=(1/6)*y1+(1/6)*y2+(1/6)*y3+(1/2)*y4;
plot(x,y)
axis([0 40 -.1 1.1])
```

(b) Compute $P(7.5 \leq X \leq 30)$.

Solution.

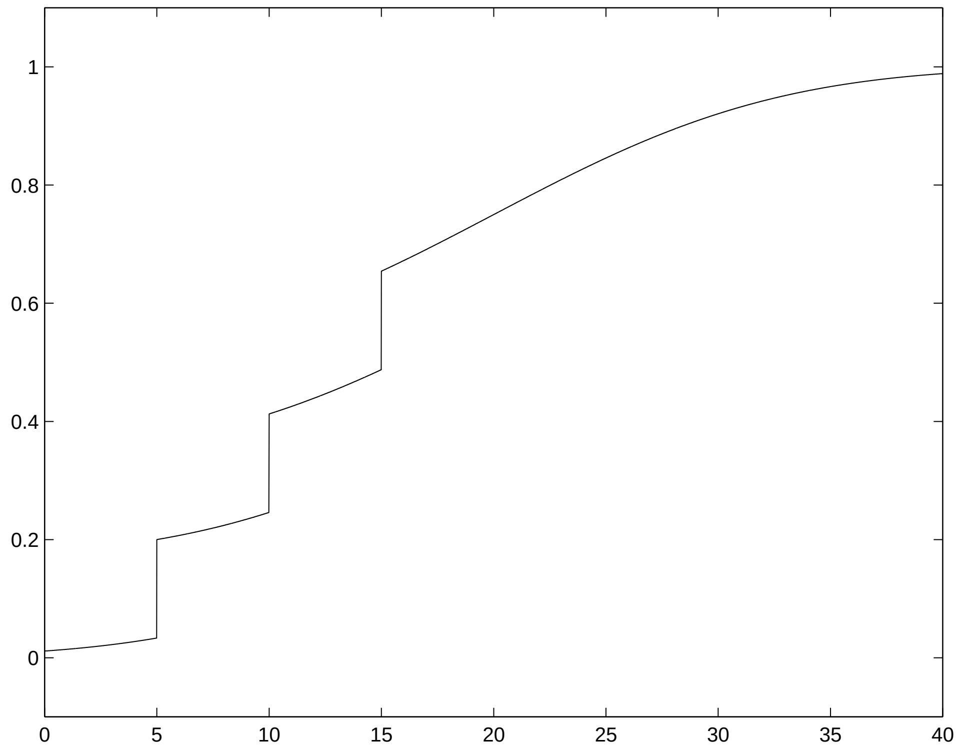
$$P(7.5 \leq X \leq 30) = (2/6) + (1/2)\int_{7.5}^{30} f(x)dx,$$

where $f(x)$ is the Gaussian$(\mu = 20, \sigma = 10)$ PDF. From page 123, this last integral is

$$\begin{aligned}
\Phi((30-20)/10) - \Phi((7.5-20)/10) &= \Phi(1) - \Phi(-1.25) \\
&= 0.8413 - (1 - \Phi(1.25)) \\
&= 0.8413 - (1 - 0.8944) = 0.7357
\end{aligned}$$

So,

$$P(7.5 \leq X \leq 30) = (1/3) + (1/2)(0.7357) = 0.701.$$



(c) Given that X is between 7.5 and 30, compute the conditional probability that $X \geq 12.5$.

Solution.

$$P(X \geq 12.5 | 7.5 \leq X \leq 30) = P(12.5 \leq X \leq 30)/(b).$$

$$\begin{aligned}
P(12.5 \leq X \leq 30) &= (1/6) + (1/2) \int_{12.5}^{30} f(x)dx \\
&= (1/6) + (1/2)(\Phi(1) - \Phi((12.5 - 20)/10)) \\
&= (1/6) + (1/2)(\Phi(1) - \Phi(-0.75)) \\
&= (1/6) + (1/2)(\Phi(1) - 1 + \Phi(0.75)) \\
&= (1/6) + (1/2)(0.8413 - 1 + 0.7734) = 0.4740
\end{aligned}$$

Therefore,

$$P(X \geq 12.5 | 7.5 \leq X \leq 30) = 0.4740/0.701 = 0.676.$$

5. This problem regards the computations of means and variances of random variables.

(a) A discrete RV X takes the values $1, 2, 3, 4, 5$ and its PMF takes the form

$$p_X(x) = Cx^2, \quad x = 1, 2, 3, 4, 5,$$

where C is a positive constant. Compute C, and then compute the mean and variance of X.

Solution. C is the reciprocal of

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$

Therefore,

$$C = 1/55$$

and the PMF values are

$$1/55, \;\; 4/55, \;\; 9/55, \;\; 16/55, \;\; 25/55,$$

respectively.

$$\mu_X = 1(1/55) + 2(4/55) + 3(9/55) + 4(16/55) + 5(25/55) = 4.0909.$$

$$E(X^2) = 1^2(1/55) + 2^2(4/55) + 3^2(9/55) + 4^2(16/55) + 5^2(25/55) = 17.8.$$

$$\sigma_X^2 = E(X^2) - \mu_X^2 = 17.8 - (4.0909)^2 = 1.0645.$$

(b) Now we take X to be a continuous RV with PDF of the form

$$f_X(x) = Cx^2, \;\; 1 \leq x \leq 5 \; (zero\; elsewhere),$$

where C is a positive constant. Compute C, and then compute the mean and variance of X.

Solution. C is the reciprocal of

$$\int_1^5 x^2 dx = 41.33,$$

which is

$$C = 0.0242.$$

$$\mu_X = \int_1^5 (0.0242) x (x^2) dx = 3.7742.$$

$$E(X^2) = \int_1^5 (0.0242) x^2 (x^2) dx = 15.1161.$$

$$\sigma_X^2 = E(X^2) - \mu_X^2 = 15.1161 - (3.7742)^2 = 0.872.$$