EE 3025 S2005 Homework Set #5

(due 10:10 AM Friday, March 4, 2005)

Directions: Work all 4 problems. We will grade Problem 1 and will randomly choose two of the other problems for grading. (There are just 4 problems this week due to the exam you had on February 28.)

1. The Monte Carlo integration technique, introduced in Recitation 4, works for double integrals as well. For example, to estimate

$$\int_2^4 \int_3^6 \sqrt{2x^3 + 3y^2}\,dx\,dy,$$

you could use the Matlab script

```
a=2;b=4; %outer limits
c=3;d=6; %inner limits
x=(d-c)*rand(1,50000)+c; %simulate inner vble values (x values)
y=(b-a)*rand(1,50000)+a; %simulate outer vble values (y values)
integrand = sqrt(2*x.^3+3*y.^2); %simulate values of integrand
MonteCarlo_estimate=(b-a)*(d-c)*mean(integrand)
```

(a) Using the preceding Matlab script as a template, estimate the following double integral in which you simulate the x variable with 50000 samples and simulate the y variable with 50000 samples:

$$\int_0^2 \int_0^2 \log_e(xy + 2)\,dx\,dy. \tag{1}$$

Obtain 10 different estimates from 10 different runs. Average these 10 estimates. Take this average as your final Monte Carlo estimate of the value of the integral (1). Give your final estimate rounded off to two decimal places only (i.e., there should be two decimal digits to the right of the decimal point).

(b) In this part of the problem, you are going to do a more precise analysis of what the value of the integral (1) might be. Let Δ be a small positive number such that $2/\Delta$ is a positive integer (such as $\Delta = 0.1, 0.05, 0.01$ or smaller). Suppose you were to partition the square

$$\{(x, y) : 0 \le x \le 2,\ \ 0 \le y \le 2\}$$

into $\Delta \times \Delta$ subsquares. In each subsquare, you could sample the integrand of (1) at the (x, y) point in the center of the subsquare. You could then add up these integrand samples (one for each subsquare), and multiply by Δ^2. Intuitively, the result would be a good approximation to the value of (1) if Δ is small enough. Write a Matlab script (in which Δ is a variable "`delta`") which will implement the method just described for estimating (1). Run your script for $\Delta = 0.1$ and see what you get. Then rerun your script with a smaller Δ. Keep rerunning your script, making Δ smaller each time, until you are satisfied that your integral estimate is accurate to two decimal places. Compare your final integral estimate to the Monte Carlo integral estimate from (a).

2. In this problem, you investigate the effect of nonlinear changes of variable on RV's.

(a) Let X be the RV defined in Homework Set 4, Problem 1(b). Compute the PDF $f_X(x)$ of X.

(b) Let Z have the standard Gaussian distribution. Let X be the random variable $X = Z^2$. Compute the PDF $f_X(x)$ of X.

(c) Suppose you were to execute the line of Matlab code

```
u=rand(1,50000);
```

See if you can devise one line of Matlab code involving the vector `u` which will create a vector `z` whose entries simulate 50000 values of a standard Gaussian RV. (Note: Your line of Matlab code cannot involve the Matlab function "`randn`" in any way!) Hint: Investigate the Matlab function "`erfinv`".

3. A pair of discrete RV's X, Y has the joint PMF given in Problem 4.2.1 of your textbook.

(a) Compute $P(X > Y)$ and $P(X < Y)$.

(b) Compute the marginal PMF $P^X(x)$ of X and the marginal PMF $P^Y(y)$ of Y.

(c) Determine whether or not

$$P(X = x, Y = y) = P(X = x)P(Y = y) = P^X(x)P^Y(y)$$

for all values x of X and values y of Y. (That is, you are determining whether or not X, Y are *statistically independent*.)

4. Random variables X, Y are jointly continuous with joint PDF as follows:

$$f_{X,Y}(x, y) = \begin{cases} C(y^2 - x^2)e^{-y}, & -y \le x \le y, \ \ 0 < y < \infty \\ 0, & \textit{for all other } (x, y) \end{cases}$$

(a) Find the value of the constant C.

(b) Find the marginal densities $f_X(x)$ and $f_Y(y)$. (Each marginal density computation involves an integral. To get the limits on your integrals right, you will probably need a sketch of the region in the xy-plane over which (X, Y) is distributed.)

(c) Compute $P(|X| \le Y/2)$.

Supplementary Problems: (not to hand in) From the textbook, you can try Problems 4.1.3, 4.3.5, 4.4.2, 4.5.4.