EE 3025 S2005 Homework Set #6 Solutions

Mr. Msechu is grading Problem 1 and Problem 4(a)(b)
Mr. AlHussien is grading Problem 2(a)(b)

Solution to Problem 1.

(a) Use the script:

```
a=0;b=2;
x=(b-a)*rand(1,50000)+a;    y=(b-a)*rand(1,50000)+a;
MonteCarlo_estimate=4*mean(log(x.*y+2))
```

The averages of the runs seem to be giving about 4.24.

(b) You can use a double for loop. A faster approach is to use Matlab meshgrid command which you can learn about in Experiment 5 of Recitation 6.

```
Delta=  ;  %enter Delta value
N=2/Delta;
x=Delta/2+Delta*(0:N-1); y=Delta/2+Delta*(0:N-1);
[X,Y]=meshgrid(x,y);
Z=log(X.*Y+2);
Integral_estimate=Delta^2*sum(sum(Z))
```

$$\begin{aligned} \Delta = 0.1 \quad &\Rightarrow \quad estimate = 4.2382 \\ \Delta = 0.05 \quad &\Rightarrow \quad estimate = 4.2379 \\ \Delta = 0.02 \quad &\Rightarrow \quad estimate = 4.2378 \end{aligned}$$

The value of the integral is probably 4.24 to two decimal places.

Solution to Problem 2.

(a) X ranges from 0 to 1. If you plot the curve $x = \sin((0.5)\pi u)$ you will see that for $0 < x < 1$,

$$P(X \le x) = P(U \le (2/\pi)\mathrm{Sin}^{-1}(x)) = (2/\pi)\mathrm{Sin}^{-1}(x).$$

Differentiating with respect to x, we see that

$$f_X(x) = \begin{cases} \frac{2}{\pi\sqrt{1-x^2}}, & 0 < x < 1 \\ 0, & elsewhere \end{cases}$$

(b) X ranges from 0 to ∞. If you fix $x > 0$, you see from the plot of the curve $x = z^2$ that

$$P(X \le x) = P(-\sqrt{x} \le Z \le \sqrt{x}) = F_Z(\sqrt{x}) - F_Z(-\sqrt{x}).$$

Differentiating with respect to x, we conclude that

$$f_X(x) = \left(\frac{1}{2\sqrt{x}}\right)[f_Z(\sqrt{x}) + f_Z(-\sqrt{x})]u(x).$$

This formula is valid no matter what the density of Z is. If you plug in the Gaussian(0,1) density

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2),$$

you see that

$$f_X(x) = \left(\frac{\exp(-x/2)}{\sqrt{2\pi x}} \right) u(x).$$

Statisticians call this distribution the *chi-squared distribution with one degree of freedom.* More generally, if you sum up the squares of n independent Gaussian(0,1) RV's, the distribution that results is called the *chi-squared distribution with n degrees of freedom.* The chi-squared distributions are important in variance estimation. I may talk about these distributions in the "statistics" segment of the course, which is what is coming up next in 3025 after Chapters 4-5.

(c) If you study Example 2 of Recitation 6, you get some hints on what to do. The answer is

```
z=sqrt(2)*erfinv(2*u-1);
```

For more details, notice that from Example 4 of Recitation 3 we have, for a standard Gaussian Z:

$$F_Z(z) = P(Z \leq z) = (1/2) * \text{erf}(z/\sqrt{2}) + (1/2).$$

Therefore, the transformation is

$$U = F_Z(Z) = (1/2) * \text{erf}(Z) + 1/2.$$

Solving for Z in terms of U:

$$Z = \sqrt{2}\text{erfinv}(2U - 1)$$

Solution to Problem 3.

(a) The joint PMF table is

	$Y=1$	$Y=3$
$X=1$	$1/28$	$3/28$
$X=2$	$2/28$	$6/28$
$X=4$	$4/28$	$12/28$

So:

$$P(X > Y) = P^{X,Y}(2,1) + P^{X,Y}(4,1) + P^{X,Y}(4,3) = 18/28.$$

$$P(X = Y) = P^{X,Y}(1,1) = 1/28.$$

$$P(X < Y) = 1 - P(X = Y) - 18/28 = 9/28.$$

(b) The $P^X(x)$'s are the row sums:

$$P^X(1) = 4/28, \quad P^X(2) = 8/28, \quad P^X(4) = 16/28.$$

The $P^Y(y)$'s are the col sums:

$$P^Y(1) = 7/28, \quad P^Y(3) = 21/28.$$

(c) Put the row sums and col sums as headers in the array:

$$\begin{array}{c c} & \begin{array}{cc} 1/4 & 3/4 \end{array} \\ \begin{array}{c} 1/7 \\ 2/7 \\ 4/7 \end{array} & \begin{pmatrix} 1/28 & 3/28 \\ 2/28 & 6/28 \\ 4/28 & 12/28 \end{pmatrix} \end{array}$$

Each of the 6 entries is the product of the row and col header for that entry. Conclusion:X, Y are independent. Alternately, observe that the 6 points where the joint PMF is positive form a Cartesian product set and the joint PMF factors over this set as a function of x $(x/28)$ alone times a function of y alone (y). Therefore, independence must hold.

Solution to Problem 4.

(a) Let R be the region where the joint density is positive.

$$\int\int_R (y^2 - x^2)e^{-y}dxdy = \int_0^\infty \int_{-y}^{y} (y^2 - x^2)e^{-y}dxdy = 8.$$

Therefore, $C = 1/8$.

(b) For $y > 0$,

$$f_Y(y) = \int_{-y}^{y} (1/8)(y^2 - x^2)e^{-y}dx = (1/6)y^3e^{-y}.$$

So,

$$f_Y(y) = (1/6)y^3e^{-y}u(y).$$

For all x on the real line,

$$f_X(x) = \int_{|x|}^{\infty} (1/8)(y^2 - x^2)e^{-y}dy = (1/4)(1 + |x|)e^{-|x|}.$$

(c)

$$P(|X| \leq Y/2) = \int_0^\infty \int_{-y/2}^{y/2} (1/8)(y^2 - x^2)dxdy = 11/16.$$