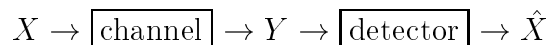


EE 3025 S2005 Homework Set #6 Solutions

Mr. AlHussien is grading Problems 1 and 5

Mr. Msechu is grading Problem 3

1. Go to Example 10 of the Recitation 6 instructions. You are going to use the discrete channel given there. The PMF of the channel input RV X is specified. From this, you can compute the joint PMF of (X, Y) where RV Y is the channel output in response to input X . In this problem, you are going to simulate the performance of every possible detector that can be inserted at the receiving end of the channel according to the following block diagram:



Each detector takes the form

$$Y = 0 \Rightarrow \hat{X} = a$$

$$Y = 1 \Rightarrow \hat{X} = b$$

$$Y = 2 \Rightarrow \hat{X} = c$$

where a, b, c are each chosen from the set of possible channel inputs $\{0, 1, 2\}$. There are therefore

$$3 * 3 * 3 = 27$$

different detectors that could be used. Of course, exactly one of them will give the minimum probability of error performance $P[X \neq \hat{X}]$; this is the MAP detector.

- (a) Using the technique of Section 17.2 of the class notes, determine the a, b, c values for the MAP detector and calculate the prob of error $P[X \neq \hat{X}]$ for the MAP detector.

Solution. The matrix of joint PMF values is

$$\begin{array}{ccc} 0.16 & 0.16 & 0 \\ 0 & 0.165 & 0.165 \\ 0.175 & 0 & 0.175 \end{array}$$

In the first column, the largest position is the $X = 2$ position. Therefore,

$$Y = 0 \Rightarrow \hat{X} = a = 2$$

In the second column, the largest position is the $X = 1$ position. Therefore,

$$Y = 1 \Rightarrow \hat{X} = b = 1$$

In the third column, the largest position is the $X = 2$ position. Therefore,

$$Y = 2 \Rightarrow \hat{X} = c = 2$$

The error prob would be

$$P[X \neq \hat{X}] = 1 - (0.175 + 0.165 + 0.175) = 0.4850.$$

- (b) Using the Matlab script given in Example 10 of Recitation 6, generate (using Matlab) a vector \mathbf{x} and a vector \mathbf{y} , each consisting of 50000 entries, simulating 50000 independent trials in which a channel input (value of X and entry of \mathbf{x}) is simulated and the resulting channel output (value of Y and corresponding entry of \mathbf{y}) is simulated. There is nothing for you to hand in regarding part(b) just stated, but as a result of part(b), you will have stored the vector \mathbf{x} and the vector \mathbf{y} in Matlab memory.
- (c) Write a Matlab script which will estimate the probability of error for each of the 27 possible detectors. The script should estimate the 27 error probs $P[X \neq \hat{X}]$ using the same \mathbf{x} and the same \mathbf{y} generated as the result of part(b). (That is, do not re-generate \mathbf{x} and \mathbf{y} each time you are testing a new detector.) Turn in printout of your script as part of your solution to this problem.

Solution.

```

u=rand(1,50000);
x=0*(u<0.32)+1*(u>=0.32 & u<0.65)+2*(u>=0.65);
z=floor(2*rand(1,50000));
y=rem(x+z,3);
results=[];
for a=0:2, for b=0:2, for c=0:2
xhat=a*(y==0)+b*(y==1)+c*(y==2);
v=[a b c mean(x~=xhat)];
results=[results;v];end,end,end
results %gives printout of desired columns

```

- (d) Run your script from part(c). Form a table with two columns. In the left column of your table you list the 27 possible a, b, c values. In the right column, you list the estimated prob of error $P[X \neq \hat{X}]$ corresponding to each left column entry. Turn in your table as part of your solution to this problem.

Solution.

| a | b | c | errorprob |
|--------|--------|--------|-----------|
| 0 | 0 | 0 | 0.6809 |
| 0 | 0 | 1.0000 | 0.5189 |
| 0 | 0 | 2.0000 | 0.5057 |
| 0 | 1.0000 | 0 | 0.6757 |
| 0 | 1.0000 | 1.0000 | 0.5137 |
| 0 | 1.0000 | 2.0000 | 0.5005 |
| 0 | 2.0000 | 0 | 0.8402 |
| 0 | 2.0000 | 1.0000 | 0.6782 |
| 0 | 2.0000 | 2.0000 | 0.6650 |
| 1.0000 | 0 | 0 | 0.8407 |
| 1.0000 | 0 | 1.0000 | 0.6787 |
| 1.0000 | 0 | 2.0000 | 0.6655 |
| 1.0000 | 1.0000 | 0 | 0.8355 |
| 1.0000 | 1.0000 | 1.0000 | 0.6735 |

| | | | |
|--------|--------|--------|--------|
| 1.0000 | 1.0000 | 2.0000 | 0.6603 |
| 1.0000 | 2.0000 | 0 | 1.0000 |
| 1.0000 | 2.0000 | 1.0000 | 0.8380 |
| 1.0000 | 2.0000 | 2.0000 | 0.8248 |
| 2.0000 | 0 | 0 | 0.6615 |
| 2.0000 | 0 | 1.0000 | 0.4995 |
| 2.0000 | 0 | 2.0000 | 0.4863 |
| 2.0000 | 1.0000 | 0 | 0.6563 |
| 2.0000 | 1.0000 | 1.0000 | 0.4943 |
| 2.0000 | 1.0000 | 2.0000 | 0.4811 |
| 2.0000 | 2.0000 | 0 | 0.8208 |
| 2.0000 | 2.0000 | 1.0000 | 0.6588 |
| 2.0000 | 2.0000 | 2.0000 | 0.6456 |

The $abc = 212$ detector won; the simulation gave error prob 0.4811.

- (e) Did the MAP detector in your left column give the smallest right column entry? (If it did not, this may have happened not because of an error on your part but because the MAP detector has a close competitor which beat it on this particular set of 50000 independent trials. In that case, you can re-run part(b) and part(d) to see if the MAP detector comes out best this time. Of course, if your MAP detector continues to lose out on run after run, then you've probably made an error somewhere.)

2. Let R be the triangular region in the first quadrant of the xy -plane bounded by the lines $x + y = 2$, $x = 0$, and $y = 0$. Suppose (X, Y) is a pair of RV's jointly continuously distributed in R . Suppose the joint CDF satisfies

$$F_{X,Y}(x, y) = Cx^2y^2, \quad (x, y) \in R,$$

where C is a positive constant.

- (a) Find C and find the expression for the joint PDF $f_{X,Y}(x, y)$ valid in R .

Solution. Taking the partial with respect to x and then with respect to y , you get

$$f_{X,Y} = 4Cxy$$

for (x, y) in R . You take the double integral of this over R and set it equal to 1 in order to determine C . You get $C = 3/8$. Thus, the joint PDF is

$$f_{X,Y} = (3/2)xy, \quad (x, y) \in R \text{ (zero elsewhere)}$$

- (b) Let S be the L-shaped region formed by connecting up the points

$$(0, 0), (0, 1.5), (.5, 1.5), (.5, .5), (1.5, .5), (1.5, 0), (0, 0) \quad (1)$$

with straight line segments (these straight line segments form the perimeter of S). Compute $P[(X, Y) \in S]$ by taking the appropriate linear combination of the joint

CDF evaluations at the points(1). (DO NOT USE the joint PDF to compute this probability.)

Solution. Use the formula from Example 18.4, page 27, of the Chapter 4-5 Class Notes. Denoting the CDF as just F , the desired prob is

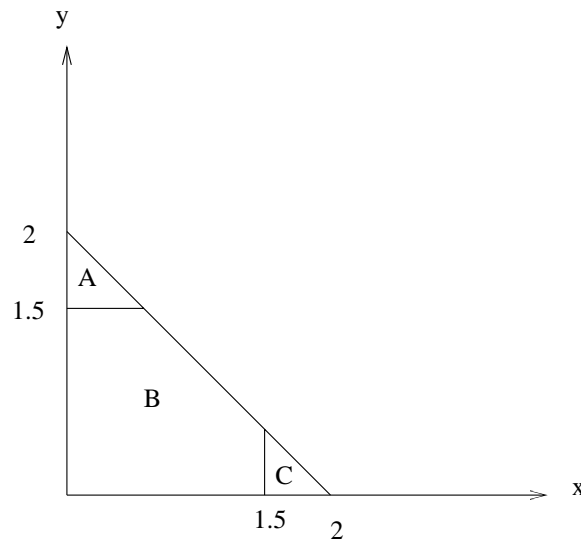
$$F(.5, 1.5) - F(.5, .5) + F(1.5, .5) - F(1.5, 0) + F(0, 0) - F(0, 1.5).$$

Only the 1st 3 terms are nonzero. From them, you get

$$(3/8)[(.5)^2(1.5)^2 - (.5)^2(.5)^2 + (1.5)^2(.5)^2] = 0.3984.$$

(c) Compute $F_{X,Y}(1.5, 1.5)$.

Solution. The answer is $P(B)$ in the figure below.



By symmetry, this is the same thing as

$$1 - 2P(C) = 1 - 2 \int_{1.5}^2 \int_0^{2-x} f_{X,Y}(x, y) dy dx = 115/128 = 0.8984.$$

3. With regard to a certain electrical circuit, we have a current I and a resistance R which are modeled as RV's that are jointly continuously distributed with joint density

$$f_{I,R}(i, r) = \begin{cases} Cir(1-r), & 0 \leq i \leq 3, 0 \leq r \leq 1 \\ 0, & elsewhere, \end{cases}$$

where C is a positive constant.

- (a) Determine whether or not the RV's I and R are statistically independent.

Solution. The region

$$\{(i, r) : 0 \leq i \leq 3, 0 \leq r \leq 1\}$$

is a Cartesian product set and you have factorization of the joint density in there as fn of i times fn of r . So I, R are independent.

(b) Compute $E[IR]$, the expected voltage that is generated by the circuit.

Solution. By independence,

$$E[IR] = E[I]E[R].$$

We have

$$E[I] = \frac{\int_0^3 i^2 di}{\int_0^3 i di} = 2.$$

$$E[R] = \frac{\int_0^1 r^2(1-r)dr}{\int_0^1 r(1-r)dr} = 1/2.$$

Therefore,

$$E[IR] = 2(1/2) = 1.$$

(c) Compute $E[I^2R]$, the expected power that is dissipated by the circuit.

Solution.

$$E[I^2R] = E[I^2]E[R].$$

$$E[I^2] = \frac{\int_0^3 i^3 di}{\int_0^3 i di} = 9/2.$$

Therefore,

$$E[I^2R] = (9/2)(1/2) = 2.25.$$

4. Let U, V be discrete RV's satisfying

$$P[U = 2, V = 0] = 0.1$$

$$P[U = 1, V = 1] = 0.1$$

$$P[U = 3, V = 1] = 0.4$$

$$P[U = 2, V = 2] = 0.4$$

Let X be the RV $X = U + V$.

(a) Compute $P[X = 2|U = 2]$ and $P[X = 2|U = 1]$. Are X and U independent?

Solution.

$$P[X = 2|U = 2] = \frac{P[X = 2, U = 2]}{P(U = 2)} = \frac{P[U = 2, V = 0]}{P[U = 2]} = \frac{.1}{.1 + .4} = .5.$$

$$P[X = 2|U = 1] = \frac{P[X = 2, U = 1]}{P(U = 1)} = \frac{P[U = 1, V = 1]}{P[U = 1]} = \frac{.1}{.1} = 1.$$

Since these probs are different, X and U are dependent, not independent.

(b) Compute the correlation $E[XU]$.

Solution. The joint PMF array of X, U is:

$$\begin{array}{ccc} & U = 1 & U = 2 & U = 3 \\ \begin{array}{l} X = 2 \\ X = 4 \end{array} & \left(\begin{array}{ccc} .1 & .1 & 0 \\ 0 & .4 & .4 \end{array} \right) \end{array}$$

Therefore,

$$E[XU] = (1 * 2).1 + (2 * 2).1 + (4 * 2).4 + (4 * 3).4 = 8.6.$$

(c) Compute the covariance $Cov[X, U]$.

Solution. The U probs are

$$P(U = 1) = .1, \quad P(U = 2) = .5, \quad P(U = 3) = .4.$$

Therefore,

$$E[U] = [1, 2, 3] \cdot [.1, .5, .4] = 2.3.$$

The X probabilities are

$$P(X = 2) = .2, \quad P(X = 4) = .8.$$

Therefore,

$$E[X] = [2, 4] \cdot [.2, .8] = 3.6.$$

Finally,

$$Cov(X, U) = E[XU] - E[X]E[U] = 8.6 - (3.6)(2.3) = 0.32.$$

(d) Compute the correlation coefficient $\rho_{X,U}$.

Solution.

$$E[U^2] = [1, 4, 9] \cdot [.1, .5, .4] = 5.7.$$

$$E[X^2] = [4, 16] \cdot [.2, .8] = 13.6.$$

$$Var(U) = E[U^2] - E[U]^2 = 5.7 - (2.3)^2 = 0.41.$$

$$Var(X) = E[X^2] - E[X]^2 = 13.6 - (3.6)^2 = 0.64.$$

Finally,

$$\rho_{X,U} = \frac{Cov(X, U)}{\sigma_X \sigma_U} = \frac{.32}{\sqrt{0.41 * 0.64}} = 0.6247.$$

5. Let R be the triangular region in the first quadrant of the xy -plane bounded by the lines $x + y = 1$, $x = 0$, and $y = 0$. Let (X, Y) be jointly continuously distributed with joint PDF as follows:

$$f_{X,Y}(x, y) = \begin{cases} 6(1 - x - y), & (x, y) \in R \\ 0, & \text{elsewhere} \end{cases}$$

(a) Compute the correlation $E[XY]$.

Solution.

$$E[XY] = \int_0^1 \int_0^{1-x} 6xy(1 - x - y) dy dx = 1/20.$$

(b) Compute the covariance $Cov[X, Y]$.

Solution.

$$E[X] = \int_0^1 \int_0^{1-x} 6x(1-x-y) dy dx = 1/4.$$

$$E[Y] = \int_0^1 \int_0^{1-x} 6y(1-x-y) dy dx = 1/4.$$

Therefore,

$$Cov(X, Y) = E[XY] - \mu_X \mu_Y = (1/20) - (1/16) = -0.0125.$$

(c) Compute the correlation coefficient $\rho_{X,Y}$.

Solution.

$$E[X^2] = \int_0^1 \int_0^{1-x} 6x^2(1-x-y) dy dx = 1/10.$$

$$E[Y^2] = \int_0^1 \int_0^{1-x} 6y^2(1-x-y) dy dx = 1/10.$$

$$Var(X) = Var(Y) = (1/10) - (1/4)^2 = 0.0375.$$

Therefore,

$$\rho_{X,Y} = \frac{Cov(X, Y)}{\sqrt{0.0375 * 0.0375}} = -1/3.$$