EE 3025 S2005 Homework Set #7 Solutions

Mr. AlHussien is grading Problem 3

Mr. Msechu is grading Problem 1, and Problem 4(a)(b)

1. You might want to try the last experiment of Recitation 7 before trying this problem.

(a) Let Z_1, Z_2, Z_3 be independent Uniform(0,1) RV's, and define X_1, X_2, X_3 to be the RV's

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 \\ -1 & 2 & 1 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \qquad (1)$$

Use Matlab to generate vectors `x1,x2,x3` of 50000 samples each of X_1, X_2, X_3. Use these three vectors to estimate the 3×3 covariance matrix Σ_X of the RV's X_1, X_2, X_3, given by

$$\Sigma_X = \begin{bmatrix} \sigma_{1,1} & \sigma_{1,2} & \sigma_{1,3} \\ \sigma_{2,1} & \sigma_{2,2} & \sigma_{2,3} \\ \sigma_{3,1} & \sigma_{3,2} & \sigma_{3,3} \end{bmatrix},$$

where $\sigma_{i,j} \triangleq Cov(X_i, X_j)$. Turn in printout of your Matlab code used to find the estimated 3×3 covariance matrix and also print out the estimated 3×3 covariance matrix that Matlab gives you. (For help, you can look at Step 3 on page 11 of Recitation 7 and extend the estimate from two variables to three variables.)

Solution.

```
Z=rand(3,50000);
A=[3 1 -1; -1 2 1; 0 2 -2];
X=A*Z;
x1=X(1,:); x2=X(2,:); x3=X(3,:);
CX_estimate=X*X'/50000-mean(X')'*mean(X')
CX_estimate =

    0.9188   -0.1680    0.3331
   -0.1680    0.5003    0.1666
    0.3331    0.1666    0.6645
```

(b) Find by hand the exact 3×3 covariance matrix Σ_Z of the RV's Z_1, Z_2, Z_3. (This is easy to do, using the independence of the Z_i's.) Let A be the 3×3 coefficient matrix on the left side of (1). Use Matlab to compute the matrix triple product

$$A * \Sigma_Z * A^T$$

and compare this answer with your estimated 3×3 matrix from (a). Are the answers about the same? Are you surprised?

Solution.

```
CZ=eye(3,3)/12;
CX=A*CZ*A'
```

```
CX =

    0.9167   -0.1667    0.3333
   -0.1667    0.5000    0.1667
    0.3333    0.1667    0.6667
```

(c) Repeat parts (a),(b) assuming that Z_1, Z_2, Z_3 are independent Gaussian(0,1) RV's.

Solution.

```
Z=randn(3,50000);
A=[3 1 -1; -1 2 1; 0 2 -2];
X=A*Z;
CX_estimate=X*X'/50000-mean(X')'*mean(X')
CX_estimate =

   11.0217   -2.0626    4.0256
   -2.0626    6.0516    1.9568
    4.0256    1.9568    7.9618

CZ=eye(3,3);
CX=A*CZ*A'
CX =

    11    -2     4
    -2     6     2
     4     2     8
```

2. Random variables X, Y are each discrete and the set S of allowable (X, Y) pairs consists of all (i, j) in which $i \leq j$ and i and j are integers between 1 and 30, inclusively. The joint PMF is of the form

$$P^{X,Y}(i,j) = Cij, \quad (i,j) \in S \ (zero\ elsewhere)$$

The computations in this problem are kind of messy, so you can use Matlab to do them if you want.

(a) Let B be the event that $X + Y > 20$. Compute the conditional PMF

$$P(X = i|B), \quad i = 1, 2, \cdots, 30$$

and put the results as two columns of a table of the form

i	$P(X = i\|B)$
col of i values	col of $P(X = i\|B)$ values

Solution.

```
[X,Y]=ndgrid(1:30,1:30);
Z=X.*Y.*(X<=Y&X+Y>20);
PXgivenB = sum(Z')/sum(sum(Z));
table=[(1:30)' PXgivenB']
table =

    1.0000    0.0025
    2.0000    0.0054
    3.0000    0.0086
    4.0000    0.0121
    5.0000    0.0158
    6.0000    0.0198
    7.0000    0.0240
    8.0000    0.0284
    9.0000    0.0329
   10.0000    0.0376
   11.0000    0.0414
   12.0000    0.0439
   13.0000    0.0462
   14.0000    0.0480
   15.0000    0.0495
   16.0000    0.0506
   17.0000    0.0513
   18.0000    0.0515
   19.0000    0.0513
   20.0000    0.0505
   21.0000    0.0491
   22.0000    0.0472
   23.0000    0.0447
   24.0000    0.0416
   25.0000    0.0378
   26.0000    0.0334
   27.0000    0.0282
   28.0000    0.0224
   29.0000    0.0157
   30.0000    0.0083
```

(b) Use your conditional PMF from (a) to compute each of the following:

$$P(X \geq 20|B), \quad E(X|B), \quad Var(X|B).$$

Solution.

```
first_answer = sum(PXgivenB(20:30))
first_answer =
```

```
    0.3790

second_answer= PXgivenB*(1:30)'
second_answer =

   16.9496

third_answer=PXgivenB*((1:30).^2)'-second_answer^2
third_answer =

   43.5605
```

$$P(X \geq 20|B) = 0.3790, \quad E(X|B) = 16.9496, \quad Var(X|B) = 43.5605.$$

3. Random variables X, Y are jointly continuously distributed with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} Cxy, & (x,y) \in R \\ 0, & \text{elsewhere} \end{cases}$$

where R is the triangular region $R = \{(x,y) : 0 \leq y \leq x, \ 0 \leq x \leq 2\}$.

(a) Plot the conditional density of Y given $X = 3/4$. Use this conditional density to compute $P[Y \geq 3/8|X = 3/4]$ and $E[Y|X = 3/4]$.

Solution. By inspection of a plot of region R, it is clear that the cond PDF can only go from $y = 0$ to $y = 3/4$. It clearly must be a constant times y. It is easy to work out what this constant is. We conclude that

$$f_{Y|X}(y|3/4) = \begin{cases} (32/9)y, & 0 \leq y \leq 3/4 \\ 0, & elsewhere \end{cases}$$

It is easy to plot this. Finally:

$$\begin{aligned} P[Y \geq 3/8|X = 3/4] &= \int_{3/8}^{3/4} (32/9)y dy = 3/4 \\ E[Y|X = 3/4] &= \int_{0}^{3/4} y(32/9)y dy = 1/2 \end{aligned}$$

(b) Plot the conditional density of X given $Y = 1$. Compute $P[X \leq 1.5|Y = 1]$ and $\text{Var}[X|Y = 1]$.

Solution. By inspection of a plot of region R, it is clear that the cond PDF can only go from $x = 1$ to $x = 2$. It clearly must be a constant times x. It is easy to work out what this constant is. We conclude that

$$f_{X|Y}(x|1) = \begin{cases} (2/3)x, & 1 \leq x \leq 2 \\ 0, & elsewhere \end{cases}$$

It is easy to plot this. Finally:

$$\begin{aligned}
P[X \leq 1.5|Y=1] &= \int_1^{1.5} (2/3)x dx = 5/12 \\
E[X|Y=1] &= \int_1^2 x(2/3)x dx = 14/9 \\
E[X^2|Y=1] &= \int_1^2 x^2(2/3)x dx = 5/2 \\
Var[X|Y=1] &= 5/2 - (14/9)^2 = 0.0802.
\end{aligned}$$

4. You are to work this problem using the law of iterated expectation:

$$E[\phi(X)\psi(Y)] = E[\phi(X)E[\psi(Y)|X]].$$

You are not allowed to use the joint density of (X, Y). In this problem, X is continuous with density

$$f_X(x) = Cx, \quad 0 \leq x \leq 3 \ (zero\ elsewhere)$$

Given $X = x$, Y is conditionally uniformly distributed between 0 and $3 - x$.

(a) Compute $E[Y]$ by the law of iterated expectation.

Solution. It is easy to determine that $C = 2/9$. By Appendix A,

$$E[Y|X=x] = (3-x)/2.$$

$$E(Y) = E(E(Y|X)) = E((3-X)/2) = \int_0^3 ((3-x)/2)(2/9)x dx = 1/2.$$

(b) Compute the correlation $E[XY]$ by the law of iterated expectation.

Solution.

$$E(XY) = E(XE(Y|X)) = E(X(3-X)/2) = \int_0^3 x((3-x)/2)(2/9)x dx = 3/4.$$

(c) Compute $E[Y^2]$ by the law of iterated expectation. Then use this answer and the answer to part(a) to compute $Var(Y)$.

Solution. By Appendix A,

$$E(Y^2|X=x) = Var(Y|X=x) + [E(Y|X=x)]^2 = (3-x)^2/12 + (3-x)^2/4 = (3-x)^2/3.$$

Therefore,

$$E(Y^2) = E(E(Y^2|X)) = E((3-X)^2/3) = \int_0^3 ((3-x)^2/3)(2/9)x dx = 1/2.$$

$$Var(Y) = E(Y^2) - \mu_Y^2 = 1/2 - 1/4 = 1/4.$$

5. Let T_1, T_2, T_3 be independent RV's each exponentially distributed with mean 1. Compute each of the following:

(a) $P[T_1 + 2T_2 + 3T_3 > 4]$

Solution. The densities of T_1, $2T_2$, and $3T_3$ are

$$\exp(-x)u(x), (1/2)\exp(-x/2)u(x), (1/3)\exp(-x/3)u(x),$$

respectively. The density of $T = T_1 + 2T_2 + 3T_3$ is the convolution of these, which you obtain by taking the inverse Laplace transform of

$$\left(\frac{1}{s+1}\right)\left(\frac{1/2}{s+1/2}\right)\left(\frac{1/3}{s+1/3}\right).$$

Matlab gave me the following partial fraction decomposition:

```
           1          4               1
    1/2 ----- - ------- + 9/2 -------
        s + 1   2 s + 1       3 s + 1
```

Taking inverse Laplace, the PDF of T is

$$f_T(x) = (1/2)\exp(-x)u(x) - 2\exp(-x/2)u(x) + (3/2)\exp(-x/3)u(x).$$

Therefore,

$$P(T > 4) = \int_4^\infty f_T(x)dx = 0.6540.$$

(b) $P[\min(T_1, 2T_2, 3T_3) > 0.2]$

Solution. This is the same as the product

$$P(T_1 > 0.2)P(2T_2 > 0.2)P(3T_3 > 0.2),$$

which is

$$\int_{0.2}^\infty \exp(-x)dx \int_{0.2}^\infty (1/2)\exp(-x/2)dx \int_{0.2}^\infty (1/3)\exp(-x/3)dx = 0.6930.$$

(c) $P[\max(T_1, 2T_2, 3T_3) > 2]$

Solution. This is the same as

$$1 - P[\max(T_1, 2T_2, 3T_3) \le 2] = 1 - P(T_1 \le 2)P(2T_2 \le 2)P(3T_3 \le 2) = 0.7340.$$