

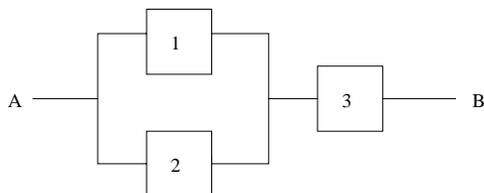
## EE 3025 S2005 Homework Set #8 Solutions

Mr. AlHussien is grading Problems 1,4

Mr. Msechu is grading Problem 5(a)

**Directions:** Work all 5 problems. We will grade Problem 1 and will randomly choose two of the other problems for grading.

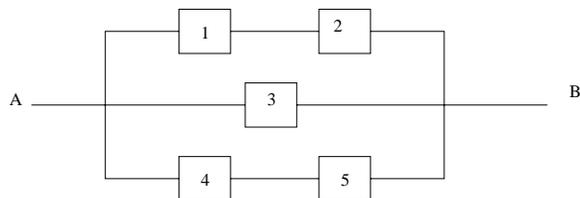
1. In the lecture of Friday, March 25, I plan to cover a method from reliability theory for finding the expected lifetime of a system assuming that the lifetimes of its independently acting subsystems are exponentially distributed with known mean lifetimes. You will use this method (combined with Matlab) to find the expected lifetime of two systems (see (a) and (b) below). Before you get started on those, you can examine the following example:



The individual components have exponentially distributed lifetimes, with the mean lifetime (in hours, say) of component  $i$  being  $(100)^i$ , for  $i = 1, 2, 3$ . Here is a Matlab script that computes the mean lifetime of the A to B connection:

```
syms t
a1=1/100; a2=1/200; a3=1/300;
p1=exp(-a1*t); p2=exp(-a2*t); p3=exp(-a3*t);
R=(1-(1-p1)*(1-p2))*p3; %the "reliability function"
lifetimeAB = int(R,0,inf)
lifetimeAB =
1545/11
```

We see that the mean lifetime of the A to B connection is  $1545/11 = 140.4545$  hours.



- (a) The system above was given in Problem 1 of Exam 1. Suppose each of the components has exponentially distributed lifetime, with mean lifetimes as follows:
- Components 1 and 2 each have mean lifetime 300 hours.
  - Component 3 has mean lifetime 250 hours.
  - Components 4 and 5 each have mean lifetime 500 hours.

Write a Matlab script and then run it to determine the expected lifetime of the A to B connection. Turn in printout of your program and result of your program run.

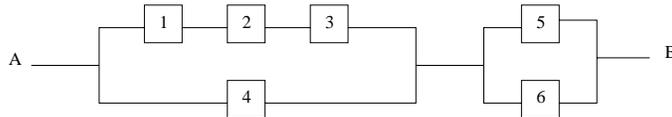
**Solution.** See Solution to Problem 1 Exam 1. Then it is easy to write the following program using the above program as a template:

```
syms t
a1=1/300; a2=1/300; a3=1/250; a4=1/500; a5=1/500;
p1=exp(-a1*t); p2=exp(-a2*t); p3=exp(-a3*t);
p4=exp(-a4*t); p5=exp(-a5*t);
R=1-(1-p1*p2)*(1-p3)*(1-p4*p5);
lifetimeAB = int(R,0,inf)
lifetimeAB =
```

8925/22

The expected lifetime of the A to B connection is  $8925/22 = 405.68$  hours.

- (b) The system below was in Problem 18 of the Exam 1 Review Problems from Recitation 5 posed at the exams web page.



Suppose components 1 through 5 have the same probabilistic characteristics as in part(a). Suppose component 6 has exponentially distributed lifetime with mean lifetime 500 hours. Write a Matlab script and then run it to determine the expected lifetime of the A to B connection. Turn in printout of your program and result of your program run.

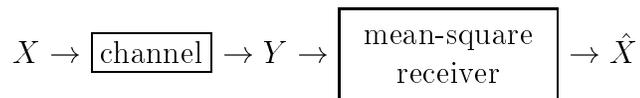
**Solution.**

```
syms t
a1=1/300; a2=1/300; a3=1/250; a4=1/500; a5=1/500; a6=1/500;
p1=exp(-a1*t); p2=exp(-a2*t); p3=exp(-a3*t);
p4=exp(-a4*t); p5=exp(-a5*t); p6=exp(-a6*t);
R=(1 - (1-p1*p2*p3)*(1-p4))*(1 - (1-p5)*(1-p6))
lifetimeAB = int(R,0,inf)
lifetimeAB =
```

217370/627

The expected lifetime of the A to B connection is  $217370/627 = 346.68$  hours.

2. In this problem, you will investigate the performance of various mean-square receivers at the receiving end of a channel as indicated in the following block diagram:



Mean-square estimation errors  $E[(X - \hat{X})^2]$  for various receivers tend to be small numbers that are hard to compare. Instead, many people evaluate the performance of a receiver by measuring the mean-square error in decibels, defined as follows:

$$\text{estimation error in decibels} \triangleq 10 \log_{10} \left[ \frac{E[X^2]}{E[(X - \hat{X})^2]} \right]. \quad (1)$$

Notice that the better the estimator  $\hat{X}$  becomes, the *smaller* the denominator becomes in (1) and therefore the *bigger* the decibel figure becomes.

For this problem, we assume that the channel is an additive noise channel. Therefore,

$$Y = X + Z,$$

where  $Z$  is the channel noise RV, assumed independent of channel input RV  $X$ . We assume that  $X$  is Gaussian with mean 1 and standard deviation 1 and that  $Z$  is Gaussian with mean 2 and standard deviation 2.

- (a) Compute the estimation error in decibels for the blind receiver

$$\hat{X} = \mu_X.$$

**Solution.** The second moment  $E[X^2]$  is 2. Also

$$E[(X - \hat{X})^2] = E[(X - \mu_X)^2] = \sigma_X^2 = 1.$$

The decibel figure is therefore

$$10 \log_{10} \frac{2}{1} = 3.01 \text{ decibels.}$$

- (b) Compute the estimation error in decibels for the correlation receiver

$$\hat{X} = \left( \frac{E[XY]}{E[Y^2]} \right) Y.$$

**Solution.** Let  $C$  be the constant in front of  $Y$ . Then it is easy to see that

$$E[Y(X - CY)] = 0.$$

(This “orthogonality relation” is also pretty clear from the geometric diagram at the top of page 48 of the Chapter 4-5 Notes.) Therefore:

$$\begin{aligned}
 E[(X - \hat{X})^2] &= E[(X - CY)^2] \\
 &= E[X(X - CY)] - CE[Y(X - CY)] \\
 &= E[X(X - CY)] - 0 \\
 &= E[X^2] - CE[XY]
 \end{aligned}$$

We have

$$E[XY] = E[X(X + Z)] = E[X^2] + E[X]E[Z] = 2 + 1 * 2 = 4.$$

$$E[Y^2] = E[X^2] + 2E[X]E[Z] + E[Z^2] = 2 + 4 + 8 = 14.$$

Therefore  $C = 4/14$ , and

$$E[(X - \hat{X})^2] = E[X^2] - CE[XY] = 2 - (4/14)4 = 6/7.$$

The decibel figure is

$$10 \log_{10} \frac{2}{6/7} = 3.68 \text{ decibels.}$$

For future reference, a nice compact way to write the answer is

$$10 \log_{10}(\csc^2 \theta),$$

where  $\theta$  is the angle such that

$$\cos \theta = \frac{E[XY]}{\sqrt{E[X^2]}\sqrt{E[Y^2]}}.$$

(c) Compute the estimation error in decibels for the minimum mean-square receiver

$$\hat{X} = E(X|Y).$$

(Hint: Since  $X, Y$  are bivariate Gaussian, Theorem 4.30 on page 194 of your textbook is applicable to part(c).)

**Solution.** By the law of iterated expectation,

$$E[(X - \hat{X})^2] = E[(X - E(X|Y))^2] = E[E[(X - E(X|Y))^2|Y]] = E[Var(X|Y)].$$

By page 194 of your text,  $Var(X|Y = y)$  does not depend on  $y$ , and is equal to

$$Var(X|Y = y) = \sigma_X^2(1 - \rho_{X,Y}^2).$$

Therefore,

$$E[(X - \hat{X})^2] = \sigma_X^2(1 - \rho_{X,Y}^2) = 1 - \rho_{X,Y}^2$$

We have

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

$$Cov(X,Y) = Cov(X, X+Z) = Cov(X,X) + Cov(X,Z) = \sigma_X^2 + 0 = 1.$$

$$\sigma_Y = \sqrt{Var(Y)} = \sqrt{\sigma_X^2 + \sigma_Z^2} = \sqrt{5}.$$

Therefore,

$$\rho_{X,Y} = \frac{1}{\sqrt{5}}.$$

We conclude that

$$E[(X - \hat{X})^2] = 1 - \rho_{X,Y}^2 = 4/5.$$

The decibel figure is therefore

$$10 \log_{10} \frac{2}{4/5} = 3.98 \text{ decibels}$$

3. Let  $X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8$  be RV's which we put in vector form as

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \\ X_8 \end{bmatrix}.$$

Using Matlab notation, suppose the mean vector is

`ones(8,1)`

and the covariance matrix is

`ones(8,8) + 2*eye(8,8)`

(a) Let  $Y$  be the random variable

$$Y = X_1 - X_2 + X_3 - X_4 + X_5 - X_6 + X_7 - X_8.$$

Find the mean and variance of  $Y$ .

**Solution.** The mean is

$$E[X_1] - E[X_2] + E[X_3] - E[X_4] + E[X_5] - E[X_6] + E[X_7] - E[X_8] = 0.$$

(Alternate terms cancel out.) The variance can be computed by the

$$AC_X A^T$$

formula, where  $A$  is a row vector. Here is a Matlab script to do this:

```

CX = ones(8,8) + 2*eye(8,8);
A = ones(1,8);
CY = A*CX*A'

```

CY =

80

The variance of  $Y$  is therefore 80.

(b) Let  $U, V$  be the random variables

$$U = \frac{X_1 + X_2 + X_3 + X_4}{4}$$

$$V = \frac{X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7 + X_8}{8}$$

Determine which of the random variables  $U, V$  has the smaller variance. (Or else conclude that they have the same variance.)

**Solution.**

```

A1=[1 1 1 1 0 0 0 0]/4;
CU=A1*CX*A1'

```

CU =

1.5000

```

A2=ones(1,8)/8;
CV=A2*CX*A2'

```

CV =

1.2500

The variance of  $U$  is 1.5 and the variance of  $V$  is 1.25. The variance of  $V$  is therefore smaller.

(c) Let  $S, T$  be the random variables

$$S = X_1 + X_2 + X_3 + X_4$$

$$T = X_5 + X_6 + X_7 + X_8$$

For what real constant  $C$  is  $E[(S - CT)^2]$  the smallest?

**Solution.** You can interpret  $CT$  as the output of a correlation receiver designed to estimate  $S$ . Therefore,

$$C = \frac{E[ST]}{E[T^2]}$$

The following Matlab script computes  $C$ :

$$\begin{aligned}
A &= \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}; \\
C\_ST &= A * C X * A'; \\
MEAN\_ST &= [4 \ 4]'; \\
R\_ST &= C\_ST + MEAN\_ST * MEAN\_ST';
\end{aligned}$$

$$\begin{aligned}
R\_ST &= \\
&\begin{bmatrix} 40 & 32 \\ 32 & 40 \end{bmatrix}
\end{aligned}$$

We see that

$$C = \frac{E[ST]}{E[T^2]} = \frac{32}{40} = 0.8.$$

4. Three-dimensional random vector  $(Y_1, Y_2, Y_3)$  has joint density  $f(y_1, y_2, y_3)$  equal to 6 in the region

$$R = \{(y_1, y_2, y_3) : 0 \leq y_1 \leq y_2 \leq y_3 \leq 1\},$$

and equal to zero elsewhere. Compute each of the three expected values  $E(Y_1)$ ,  $E(Y_2)$ ,  $E(Y_3)$ . Hint: For any RV  $\phi(Y_1, Y_2, Y_3)$ , the expected value is

$$E[\phi(Y_1, Y_2, Y_3)] = \iiint_R \phi(y_1, y_2, y_3)(6) dy_1 dy_2 dy_3.$$

**Solution.**

$$E[Y_1] = \int_0^1 \int_{y_1}^1 \int_{y_2}^1 6y_1 dy_3 dy_2 dy_1 = 1/4$$

$$E[Y_2] = \int_0^1 \int_{y_1}^1 \int_{y_2}^1 6y_2 dy_3 dy_2 dy_1 = 1/2$$

$$E[Y_3] = \int_0^1 \int_{y_1}^1 \int_{y_2}^1 6y_3 dy_3 dy_2 dy_1 = 3/4$$

5. This problem concerns the construction of confidence intervals. In each part(a),(b), you will determine how many independent random samples you need to take from a probability distribution with unknown mean  $\mu$  so that a confidence interval for estimating  $\mu$  can be constructed with a given level of confidence and a given width.

- (a) Denote your independent random samples from the fixed probability distribution by

$$X_1, X_2, \dots, X_n.$$

The mean  $\mu$  of the fixed probability distribution is unknown. However, it is known that the fixed probability distribution is Gaussian with known standard deviation  $\sigma = 2$ . Let  $\bar{X}$  be the sample mean, defined by

$$\bar{X} \triangleq \frac{X_1 + X_2 + \dots + X_n}{n}.$$

Determine the smallest number of samples  $n$  you can so that

$$P\left(\bar{X} - 0.05 \leq \mu \leq \bar{X} + 0.05\right) \geq 0.80.$$

(Motivation: You are constructing an 80% confidence interval. The 0.05 was chosen in an attempt to make us confident that we are getting one decimal place accuracy in estimating  $\mu$  using  $\bar{X}$ .)

**Solution.** The confidence interval is of the form

$$\left[\bar{X}_n - \frac{k\sigma}{\sqrt{n}}, \bar{X}_n + \frac{k\sigma}{\sqrt{n}}\right],$$

where  $k$  is chosen so that

$$P[-k \leq Z \leq k] = 0.80,$$

$Z$  being a Gaussian(0,1) RV. We have

$$P[-k \leq Z \leq k] = 2\Phi(k) - 1 = 0.80,$$

and so

$$\Phi(k) = 0.90,$$

and

$$k = 1.2816.$$

Now set

$$\frac{(1.2816)\sigma}{\sqrt{n}} = \frac{(1.2816)2}{\sqrt{n}} = 0.05,$$

and solve for  $n$ . You get

$$n = 2628.$$

- (b) (This part of the problem is *optional*—you need not work it.) Again, denote your independent random samples from the fixed probability distribution by

$$X_1, X_2, \dots, X_n.$$

The mean  $\mu$  of the fixed probability distribution is unknown. The fixed probability distribution is unknown and its standard deviation is unknown. However, it is known that the fixed probability distribution is a discrete probability distribution over the two binary values 0,1. Using these given facts, determine the smallest number of samples  $n$  you can so that

$$P\left(\bar{X} - 0.05 \leq \mu \leq \bar{X} + 0.05\right) \geq 0.80.$$

(Hint: You will be using Chebyshev's Inequality and you will also have to use somehow the assumption that you are taking the samples from the set  $\{0, 1\}$ .)

**Solution.** The sampling distribution assigns prob  $p$  to 1 and therefore prob  $1 - p$  to 0, where  $p$  is unknown and can be anything between 0 and 1. The mean  $\mu$  is  $p$ . We attempt a confidence interval for  $\mu = p$  of the form

$$[\bar{X}_n - \frac{k\sigma}{\sqrt{n}}, \bar{X}_n + \frac{k\sigma}{\sqrt{n}}],$$

where we use Chebyshev's inequality to assert that

$$1 - \frac{1}{k^2} = 0.80.$$

We see that

$$k = \sqrt{5}.$$

Plugging this  $k$  value and  $\sigma = \sqrt{p(1-p)}$  in the conf interval endpoints, you get the interval

$$[\bar{X}_n - \frac{\sqrt{5}\sqrt{p(1-p)}}{\sqrt{n}}, \bar{X}_n + \frac{\sqrt{5}\sqrt{p(1-p)}}{\sqrt{n}}]$$

Unfortunately, the endpoints depend on  $p$ , but if we take the largest of these intervals as  $p$  varies as our confidence interval, we will surely achieve at least 80% confidence as well. The largest interval occurs for  $p = 1/2$ . We therefore take our confidence interval to be of the form

$$[\bar{X}_n - \frac{\sqrt{5}\sqrt{1/4}}{\sqrt{n}}, \bar{X}_n + \frac{\sqrt{5}\sqrt{1/4}}{\sqrt{n}}]$$

To find the sample size  $n$ , solve the equation

$$\frac{\sqrt{5}\sqrt{1/4}}{\sqrt{n}} = 0.05.$$

You get  $n = 500$ .