EE 3025 S2005 Homework Set #9

(due 10:10 AM Friday, April 15, 2005)

Directions: Work all 5 problems. We will grade Problem 1 and will randomly choose two of the other problems for grading.

1. In this Matlab problem, you have an additive noise channel which operates at discrete instants of time

$$i = 1, 2, 3, 4, \cdots$$

in such a way that the characteristics of the channel noise keeps changing with time. (In the real world, such as in a wireless communication system, the characteristics of your channel can keep changing with time in an unpredictable way.) Let us model the inputs to our channel as an INFINITELY LONG sequence of independent Gaussian(0,1) RV's

$$X_i, \quad i = 1, 2, 3, 4, \cdots \tag{1}$$

Thus, RV X_i for fixed i denotes the random channel input at time i. (In the language of Chapter 10, the sequence (1) is called the *channel input process*.) Let us model the channel noise as an INFINITELY LONG sequence of independent RV's

$$Z_i, \quad i = 1, 2, 3, 4, \cdots \tag{2}$$

where Z_i is Gaussian with mean 0 and variance 1/4 for i odd and Z_i is Gaussian with mean 0 and variance 1 for i even. (In the language of Chapter 10, the sequence (2) is called the *channel noise process*.) The so-called *channel output process* is the INFINITELY LONG sequence of RV's

$$Y_i, \quad i = 1, 2, 3, 4, \cdots$$

in which

$$Y_i = X_i + Z_i \tag{3}$$

at each time i. (This equation says that the channel output Y_i at time i is equal to the sum of the channel input X_i at time i and the channel noise RV Z_i at time i.) The purpose of this Matlab problem is for you to design a *time-invariant correlation receiver*. To design such a correlation receiver, you choose a real constant C and then, for each time i, the correlation receiver estimate for input X_i will be of the form

$$\hat{X}_i = CY_i. \tag{4}$$

Notice that in equation (4), you are not allowed the change the value of C with time, the reason being that you, as the user of this channel, may be ignorant about how the channel noise changes with time, and so it would not be possible for you to vary C with time in order to take advantage of the times when the channel noise variance is smaller than at other times.

The following block diagram summarizes what is going on as a function of time i:

$$X_i \to \boxed{\text{channel}} \to Y_i = X_i + Z_i \to \boxed{\text{corr rec}} \to \hat{X}_i = CY_i$$

You are going to use the following methodology for finding C:

- Use Matlab function "randn" to simulate 50000 channel inputs from time $i = 1$ thru time $i = 50000$. Denote this sequence of 50000 simulated inputs by (x_i).
- Use Matlab function "randn" to simulate the channel noise from time $i = 1$ thru time $i = 50000$. Denote this sequence of 50000 simulated noise samples by (z_i).
- Combine (x_i) and (z_i) according to the channel operation equation (3) to obtain 50000 simulated channel outputs (y_i) from time $i = 1$ thru time $i = 50000$.
- For a fixed C in (4), obtain the 50000 simulated corr rec outputs $(\hat{x}_i)$ from time $i = 1$ thru time $i = 50000$.
- Try different C's until you find a C for which

$$\frac{\sum_{i=1}^{50000}(x_i - \hat{x}_i)^2}{50000} = \frac{\sum_{i=1}^{50000}(x_i - Cy_i)^2}{50000}$$

is as small as possible.

(a) Write and turn in a Matlab script which stores in Matlab memory the simulated sequences (x_i), (z_i), (y_i) (each of length 50000).

(b) Write and turn in a Matlab script which generates a plot of

$$\frac{\sum_{i=1}^{50000}(x_i - Cy_i)^2}{50000}$$

as a function of C. Turn in the resulting plot. (By trying part(b) a few times over different ranges of C, you can figure out what range of C it would be appropriate to use as the horizontal axis for your plot.)

(c) What value of C (roughly) seems to be making your plot in (b) reach its minimum value?

(d) This part of the problem is optional, because you may not yet feel comfortable with what I'm going to ask you to do. (However, by the end of the course, you'd hopefully be able to come back and answer this part.) What would be the "ideal" theoretical value to take for C? This would be the value of C for which

$$\lim_{n\to\infty}\left\{\frac{\sum_{i=1}^{n} E[(X_i - CY_i)^2]}{n}\right\}$$

is a minimum.

2. This problem goes a little further with Chapter 9 mean square estimation theory than we were able to do before Exam 2. Let continuous RV Q be uniformly distributed between 0 and 1. Given $Q = q$, let RV K be the number of heads on three flips of a coin with $P[H] = q$.

(a) Find the two constants A, B such that

$$\hat{Q} = AK + B$$

is the straight line mean square estimator for Q based on K. In other words, A, B are chosen so that

$$E[(Q - \hat{Q})^2]$$

is a minimum. Then, estimate the decibel figure

$$10 \log_{10} \frac{E[Q^2]}{E[(Q - \hat{Q})^2]} \tag{5}$$

via Matlab using 50000 simulated values for the pair (Q, K) and the resulting 50000 simulated values for $\hat{Q}$. Turn in your Matlab program and the result of your program run.

(b) Note that Q is continuous and K is discrete. We have not yet had much experience with such a "mixed" case in mean square estimation theory. Therefore, I will tell you how $E[Q|K = k]$ is computed. You compute it via the formula

$$E[Q|K = k] = \frac{\int_0^1 qP[K = k|Q = q]f_Q(q)dq}{\int_0^1 P[K = k|Q = q]f_Q(q)dq}.$$

Use the preceding formula to compute each of the 4 values

$$E[Q|K = 0], \quad E[Q|K = 1], \quad E[Q|K = 2], \quad E[Q|K = 3].$$

Then, estimate the decibel figure

$$10 \log_{10} \frac{E[Q^2]}{E[(Q - E[Q|K])^2]} \tag{6}$$

via Matlab using 50000 simulated values for the pair (Q, K) and the resulting 50000 simulated values for $E[Q|K]$. Turn in your Matlab program and the result of your program run. (Note: The decibel figure (6) should be *greater than or equal to* the decibel figure (5).)

3. The delay in a communication system is modeled as an exponentially distributed RV T, with mean 1. We assume that the system is a pure delay system. This means that if the input to the system is the deterministic signal $\phi(t)$, then the output from the system is the random signal $\phi(t - T)$.

(a) Suppose the input to the pure delay system is the unit step function $u(t)$. Then the output is the random signal

$$Y(t) = u(t - T), \quad t \geq 0$$

It is easy to see that for each fixed $t \geq 0$, the 1-D cross-section random variable $Y(t)$ is a discrete random variable taking the binary values $0, 1$. Compute $P[Y(t) = 0]$ and $P[Y(t) = 1]$ (these will be functions of t).

(b) Suppose the input to the pure delay system is now the ramp function $r(t) = tu(t)$. Then the output is the random signal

$$Y(t) = r(t - T), \quad t \geq 0$$

For each fixed $t \geq 0$, the 1-D cross-section random variable $Y(t)$ is a mixed random variable, taking the discrete value 0, while being continuously distributed over the interval $(0, t]$. This implies that the PDF $f_{Y(t)}(y)$ of the RV $Y(t)$ is expressible in the form

$$f_{Y(t)}(y) = P[Y(t) = 0]\delta(y) + P[Y(t) > 0]f_{Y(t)}(y|Y(t) > 0)$$

where $f_{Y(t)}(y|Y(t) > 0)$ is the conditional PDF of $Y(t)$ given $Y(t) > 0$. Evaluate $P[Y(t) = 0]$ as a function of t and find an expression for $f_{Y(t)}(y|Y(t) > 0)$.

4. Let A and B be independent RV's satisfying

$$P[A = \pm 1] = 1/2, \quad P[B = \pm 2] = 1/2$$

We consider the random process

$$X(t) = A\sin(\pi t/2) + B\cos(\pi t/2), \; -\infty < t < \infty$$

(a) Use Matlab to plot the four realizations of the process $X(t)$ on the same set of axes. Plot for $0 \leq t \leq 4$ only.

(b) Compute the mean function $E[X(t)]$ of the process $X(t)$.

(c) Compute $E[X(t)X(t+1)]$ for $t = 0, 1, 2, 3$. Are these four values the same or not?

(d) Compute $E[X(t)X(t+2)]$ for $t = 0, 1$. Are these two values the same or not?

Supplementary Problems: (not to hand in) From the textbook, you can try Problems 9.2.7, 10.2.3, 10.4.1, 10.5.1.