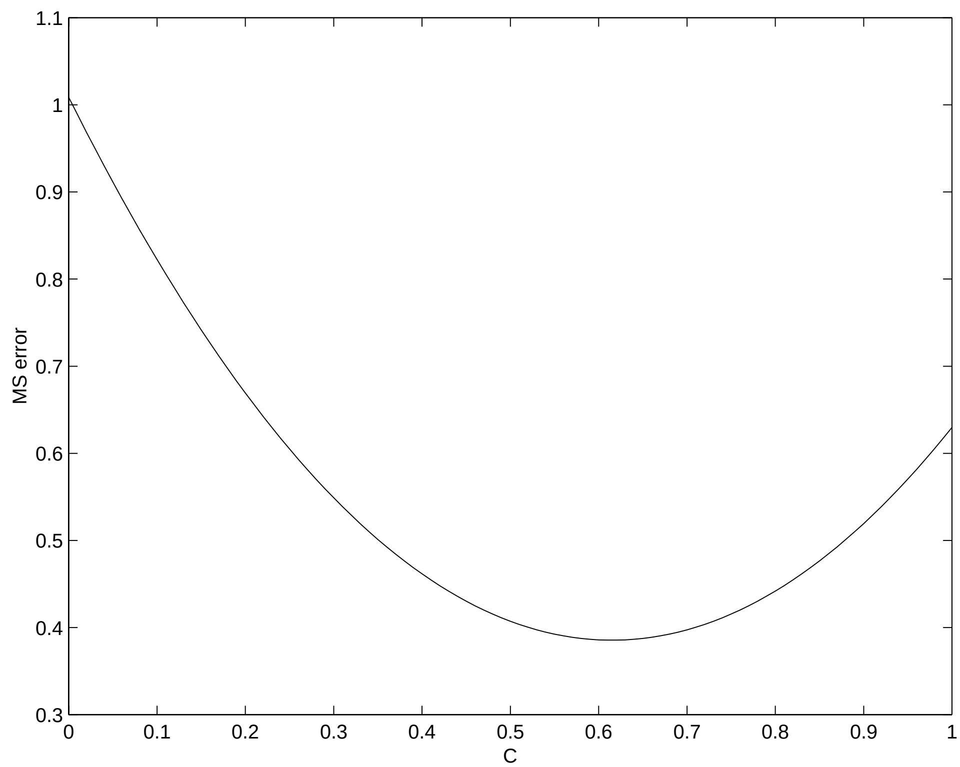
EE 3025 S2005 Homework Set #9 Solutions

Mr. AlHussien is grading Problem 2
Mr. Msechu is grading Problems 1,3

Solution to Problem 1.

Solution to (a).
```
x=randn(1,50000);
z(1:2:49999)=0.5*randn(1,25000);
z(2:2:50000)=randn(1,25000);
y=x+z;
```

Solution to (b).
```
C=0:0.01:1;
for i=1:length(C)
c=C(i);
MSerror(i)=mean((x-c*y).^2);
end
plot(C,MSerror)
```



Solution to (c).
```
[a,b]=min(MSerror); C(b)
ans =

    0.6100
```

The min occurs at about $C = .6$.

Optional Solution to (d). Pick C so that

$$E[(X_1 - CY_1)^2] + E[(X_1 - CY_2)^2]$$

is a minimum. Differentiating with respect to C and setting derivative equal to zero, you get

$$E[(X_1 - CY_1)Y_1] + E[(X_2 - CY_2)Y_2] = 0.$$

It follows that

$$C = \frac{E[X_1Y_1] + E[X_2Y_2]}{E[Y_1]^2 + E[Y_2]^2}.$$

Solving, one obtains

$$C = 8/13 = 0.6154.$$

Solution to Problem 2.

Solution to (a). From the orthogonality relations,

$$\begin{aligned} E[(Q - \hat{Q})K] &= 0 \\ E[Q - \hat{Q}] &= 0 \end{aligned}$$

one sees that the system of equations to be solved for A, B is:

$$\begin{aligned} E[K^2]A + E[K]B &= E[QK] \\ E[K]A + B &= E[Q] \end{aligned}$$

Using the law of the iterated expectation,

$$\begin{aligned} E[K] &= E[E[K|Q]] = E[3Q] = 3/2 \\ E[K^2] &= E[E[K^2|Q]] = E[3Q(1-Q) + (3Q)^2] = 7/2 \\ E[KQ] &= E[E[KQ|Q]] = E[QE[K|Q]] = E[3Q^2] = 1 \end{aligned}$$

Thus, the equations to be solved are

$$\begin{aligned} (7/2)A + (3/2)B &= 1 \\ (3/2)A + B &= 1/2 \end{aligned}$$

Solving the system, one gets $A = B = 1/5$.

```
q=rand(1,50000);
Q=[q;q;q];
k=sum(rand(3,50000)<Q);
qhat=(k+1)/5;
decibels=10*log10((1/3)/mean((q-qhat).^2))
decibels =

   10.0170
```

Our estimate for the estimation error in decibels is about 10 decibels. For fun, let us compute the actual decibel figure. From page 2 of Recitation 11, it is

$$10\log_{10}\left[\frac{(1/3)}{E}\right],$$

where

$$E = \sigma_Q^2\left(1 - \frac{Cov(Q,K)^2}{\sigma_Q^2\sigma_K^2}\right).$$

Plug in

$$Cov(Q,K) = 1/4, \ \ \sigma_Q^2 = 1/12, \ \ \sigma_K^2 = 5/4.$$

You get $E = 1/30$. The exact decibel figure for the MS estimation error is

$$10\log_{10} 10 = 10 \ \ decibels.$$

Solution to (b).

$$E[Q|K=k] = \frac{\int_0^1 q^{k+1}(1-q)^{3-k}dq}{\int_0^1 q^k(1-q)^{3-k}dq} = (k+1)/5.$$

This is the same as the straight line estimator. The MS estimation error is therefore 10 decibels.

Solution to Problem 3.

Solution to (a). $Y(t)$ is discrete taking the values 0 and 1. We have

$$P[Y(t)=0] = P[T>t] = \int_t^\infty \exp(-x)dx = \exp(-t).$$

$$P[Y(t)=1] = 1-\exp(-t).$$

Solution to (b). The density of $W = T - t$ is

$$f_W(w) = \exp(w-t)u(t-w).$$

Note that

$$Y(t) = r(W).$$

The ramp function is similar to the hard limiter treated in Problem 1.2 of the Chapter 4-5 Solved Problems. So, looking at the solution of that solved problem, we know what the density of $Y(t)$ is going to be: The ramp function preserves all positive W values and that part of the W density is left alone and becomes part of the $Y(t)$ density; the ramp function converts all negative W values into 0, and so the left tail of the W density gets converted to a delta function $A\delta(y)$ in the $Y(t)$ density, of height A equal to the left tail probability $P[W<0] = \exp(-t)$ of the W density. The answer is therefore:

$$f_{Y(t)}(y) = \exp(-t)\delta(y) + \exp(y-t)[u(y) - u(y-t)].$$

Solution to Problem 4. Let A and B be independent RV's satisfying

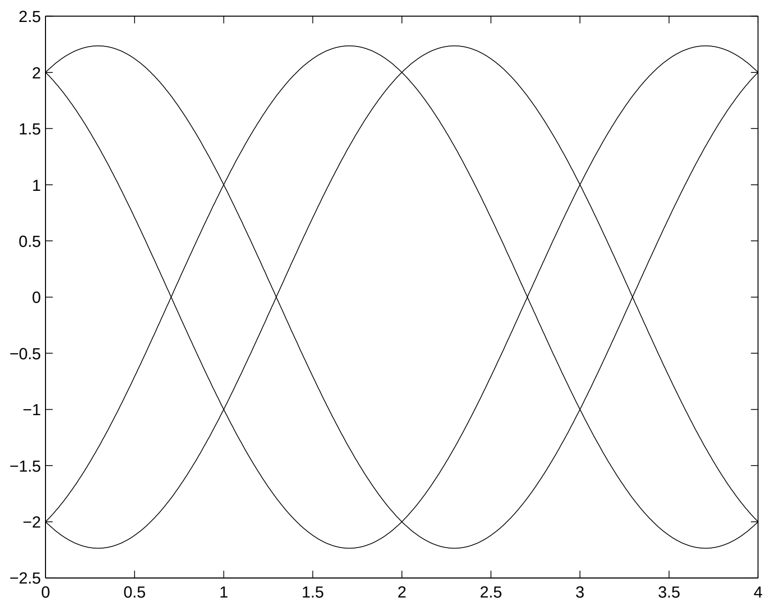
$$P[A = \pm 1] = 1/2, \quad P[B = \pm 2] = 1/2$$

We consider the random process

$$X(t) = A\sin(\pi t/2) + B\cos(\pi t/2), \ -\infty < t < \infty$$

Solution to (a).

```
t=0:.01:4;
A=1; B=2;
x1=A*sin(pi*t/2)+B*cos(pi*t/2);
A=1; B=-2;
x2=A*sin(pi*t/2)+B*cos(pi*t/2);
A=-1; B=2;
x3=A*sin(pi*t/2)+B*cos(pi*t/2);
A=-1; B=-2;
x4=A*sin(pi*t/2)+B*cos(pi*t/2);
plot(t,x1,t,x2,t,x3,t,x4)
```



Solution to (b).

$$\mu_X(t) = E[X(t)] = E[A]\sin(\pi t/2) + E[B]\cos(\pi t/2) = 0 + 0 = 0.$$

Solution to (c).

$$\begin{aligned} E[X(t)X(t+1)] &= E[A^2]\sin(\pi t/2)\sin(\pi(t+1)/2) + E[B^2]\cos(\pi t/2)\cos(\pi(t+1)/2) \\ &+ E[AB](\text{stuff}) \end{aligned}$$

Plugging in $E[A^2] = 1$, $E[B^2] = 4$, and $E[AB] = E[A]E[B] = 0$, we see that

$$\begin{aligned}
E[X(t)X(t+1)] &= \sin(\pi t/2)\sin(\pi(t+1)/2) + 4\cos(\pi t/2)\cos(\pi(t+1)/2) \\
E[X(0)X(1)] &= 0 \\
E[X(1)X(2)] &= 0 \\
E[X(2)X(3)] &= 0 \\
E[X(3)X(4)] &= 0
\end{aligned}$$

(because the sine function vanishes at even integer multiples of $\pi/2$ and the cosine function vanishes at odd integer multiples of $\pi/2$). All four correlations are equal to zero.

Soluton to (d).

$$\begin{aligned}
E[X(t)X(t+2)] &= \sin(\pi t/2)\sin(\pi(t+2)/2) + 4\cos(\pi t/2)\cos(\pi(t+2)/2) \\
E[X(0)X(2)] &= -4 \\
E[X(1)X(3)] &= -1
\end{aligned}$$

Notice that the time separation is 2 seconds in each case. Since these two autocorrelations are not the same, the process is not WSS.