11 Rec 11:Mean-Square Estimation/Random Process Introduction

Directions: Your instructor will spend the the first 40 minutes of the recitation period working some review problems and going over one or more Matlab experiments in the following. During the last 10 minutes of recitation, your proctor will give you a "Lab Form" that your recitation team completes, signs, and turns in. See the last page for an indication of what you will be asked to do on the Lab Form.

Due to time limitations, only a part of the following can be covered during the recitation period. However, you might want in the future to try some of the uncovered experiments on your own. They could give skills useful on some future homework problems and could lend insight into your understanding of the course from an experimental point of view.

This Week's Topics.

- Straight Line Receiver Versus Correlation Receiver
- Least Squares Straight Line Fitting
- Realizations of Random Processes
- 1-D/2-D Cross-Sections of a Process
- Introduction to Poisson Process

11.1 Exp 1: Straight Line Receiver Versus Correlation Receiver

In the Lecture 21 Notes, the correlation receiver is developed, and you did a little bit with it in a previous lab report. In Section 24.2 of the Lecture 24 Notes, the straight line receiver is developed. It provides a smaller mean square estimation error performance than the correlation receiver. This fact can be demonstrated with Matlab simulations, and we will lead you toward this in the present experiment.

Suppose in the block diagram

$$X \rightarrow \boxed{\text{channel}} \rightarrow Y \rightarrow \boxed{\text{receiver}} \rightarrow \hat{X} = AY + B$$

the estimator is the *straight-line receiver*, which is the receiver of the form $\hat{X} = AY + B$ which minimizes mean-square estimation error $E[(X - \hat{X})^2]$ among all receivers of the straight line form. Mean-square estimation theory tells us that there are two ways to find the constants A and B:

Method 1:

$$\begin{aligned} A &= \rho_{X,Y}\sigma_X/\sigma_Y \\ B &= \mu_X - A\mu_Y \end{aligned}$$

Method 2: Solve

$$\begin{aligned} AE[Y^2] + BE[Y] &= E[XY] \\ AE[Y] + B &= E[X] \end{aligned}$$

Whichever way you solve the problem, you can solve the problem if you know the vector of means

$$\begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}$$

along with the covariance matrix

$$\begin{bmatrix} \sigma_X^2 & \sigma_{X,Y} \\ \sigma_{X,Y} & \sigma_Y^2 \end{bmatrix}.$$

Example 1. Let the vector of means be

$$\begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix} = \begin{bmatrix} -1 \\ 2.5 \end{bmatrix}$$

and let the covariance matrix be

$$\begin{bmatrix} \sigma_X^2 & \sigma_{X,Y} \\ \sigma_{X,Y} & \sigma_Y^2 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 7 \end{bmatrix}.$$

Use Method 1 to find the optimum choice of A, B in the straight line estimator $\hat{X} = AY + B$.

Example 2. For the same (X, Y) as in Example 1, now use Method 2 to find the optimum choice of A, B in the straight line estimator $\hat{X} = AY + B$. (Write the system of two equations in Method 2 in matrix form, and then use the inverse of the 2×2 coefficient matrix to solve the system.) Obviously, you should get the same answers for A, B as you got in Example 2.

Example 3. The straight line receiver is better than the correlation receiver in the following sense: it yields mean square estimation error $E[(X - \hat{X})^2]$ less than or equal to the mean square estimation error yielded by the correlation receiver $\hat{X} = CY$. In this example, you obtain Matlab verification of this fact. In a future lecture or in the class notes, I will prove that the mean square estimation error for the correlation receiver is given by the formula

$$corr\ receiver\ estimation\ error\ = E[X^2]\left(1 - \frac{E[XY]^2}{E[X^2]E[Y^2]}\right) \tag{1}$$

I will also prove that the mean square estimation error $E[(X - \hat{X})^2]$ for the straight line receiver is given by the formula

$$straight\ line\ receiver\ estimation\ error\ = \sigma_X^2\left(1 - \frac{Cov(X,Y)^2}{\sigma_X^2\sigma_Y^2}\right) \tag{2}$$

Run the Matlab script:

```
x=randn(1,50000)+1;
z=2*randn(1,50000)+2;
y=x+z;
```

You have now stored in Matlab memory a vector `x` of 50000 simulated values of a random variable X and a vector `y` of 50000 simulated values of a random variable Y. The random pair (X, Y) may be viewed as the random input and output, respectively, from a Gaussian additive noise channel. Use the vectors `x,y` to obtain estimates of the two figures (1)-(2). Your instructor will give you some hints on obtaining the estimates. (For example,

```
mean(x.^2)
```

estimates $E[X^2]$, and `var(x)` estimates σ_X^2.) See if your estimate for (2) is less than your estimate for (1). Do the experiment again to see if your estimates "hold true". Now pick a different correlated random pair (X, Y) that you can do the experiment on. (Hint: In the Matlab script above for generating vectors `x,y`, replace each "randn" by "rand".)

11.2 Exp 2: Least Squares Straight Line Fitting

"Least Squares straight line fitting to data" is something you typically do in a freshman or sophomore physics lab: you plot a bunch of (x, y) data points that you obtain from some experiment and then you try to pass a straight line through these points. It is interesting to note that the theory of straight line receivers gives us a mechanism for solving the "Least Squares straight line fitting" problem. You will become aware of this fact via this experiment.

Let n be a positive integer. Suppose we are given a vector of n observations $x = (x_i : i = 1, 2, \cdots, n)$ and a vector of n observations $y = (y_i : i = 1, 2, \cdots, n)$. We will call (x_i) the "x-data" and we will call (y_i) the "y-data." In many science and engineering applications you have to find a least-squares straight-line fit of the y-data to the x-data. This means you find a straight line $x = Ay + B$ such that

$$\sum_{i=1}^{n} (x_i - \{Ay_i + B\})^2$$

is a minimum. Taking the partial derivatives with respect to A and B and setting them equal to zero, we see that A and B are found by solving the following equations (written in Matlab syntax):

$$\begin{aligned} A * \text{mean}(y.\hat{}2) + B * \text{mean}(y) &= \text{mean}(x. * y) \\ A * \text{mean}(y) + B &= \text{mean}(x) \end{aligned} \tag{3}$$

- *Example 4.* Run the following Matlab code in order to store in Matlab memory a data vector `x` and a data vector `y`, each consisting of 10000 samples:

  ```
  u=randn(1,10000);
  v=randn(1,10000);
  x=u-3*v;
  y=2*u+v;
  ```

Now run some more lines of Matlab code that will compute the constants A, B such that the straight line $x = Ay + b$ is the best mean-square straight line fit of the y-data given by `y` to the x-data given by `x`.

- *Example 5.* Let U, V be independent standard Gaussian RV's. Let X, Y be the dependent RV's defined by

$$\begin{aligned} X &= U - 3V \\ Y &= 2U + V \end{aligned}$$

Let $\hat{X} = AY + B$ be the minimum least-squares linear estimator of X based on Y. Find approximations to A and to B using Matlab. Hint: With `x` and `y` the pseudorandom vectors generated in Example 4, argue that the solutions A, B to the system (3) are approximately the same as the solutions to

$$\begin{aligned} AE[Y^2] + BE[Y] &= E[XY] \\ AE[Y] + B &= E[X] \end{aligned}$$

11.3 Exp 3: Realizations of Random Processes

If $(X_n : n = 1, 2, 3, \cdots)$ is a discrete-time random process (DTRP), and you observe that the component RV X_n takes the value x_n for each n, then the DT signal $x_n, n \geq 0$, is a realization of the DTRP. Similarly, if $(X(t) : t \geq 0)$ is a continuous-time random process (CTRP), and you observe that the component RV $X(t)$ takes the value $x(t)$ for each t, then the CT signal $x(t), t \geq 0$, is a realization of the CTRP. In this experiment, you plot some realizations of random processes in order to get an idea of what the realizations look like.

- *Example 6.* The following Matlab code plots four different realizations of the Bernoulli coin-flip process $(X_n : n = 1, 2, 3, \cdots)$, plotted from $n = 1$ to $n = 50$ only. (This is the process which at time n generates ± 1 depending on whether n-th flip of fair coin is heads or tails.)

```
u=rand(1,50);
x1=(u>1/2)-(u<=1/2);
subplot(2,2,1)
bar(1:50,x1,.05)
u=rand(1,50);
x2=(u>1/2)-(u<=1/2);
subplot(2,2,2)
bar(1:50,x2,.05)
u=rand(1,50);
x3=(u>1/2)-(u<=1/2);
subplot(2,2,3)
bar(1:50,x3,.05)
u=rand(1,50);
x4=(u>1/2)-(u<=1/2);
subplot(2,2,4)
bar(1:50,x4,.05)
```

Suppose you were able to plot a realization of the Bernoulli process from $n = 1$ to $n = 10000$. About how many of the lines in the plot would be pointing up and how many pointing down?

- *Example 7.* In this example, you plot some realizations of the random walk process $(Y_n : n = 0, 1, 2, \cdots)$ (i.e., the drunkard's walk) from $n = 0$ to $n = 20$. (This is the process you get by passing the Bernoulli process through a discrete-time integrator.)

```
u=rand(1,20);
x=(u>1/2)-(u<=1/2);
y1=[0 cumsum(x)];
subplot(2,2,1)
plot(0:length(y1)-1,y1)
u=rand(1,20);
x=(u>1/2)-(u<=1/2);
y2=[0 cumsum(x)];
subplot(2,2,2)
plot(0:length(y2)-1,y2)
u=rand(1,20);
x=(u>1/2)-(u<=1/2);
y3=[0 cumsum(x)];
subplot(2,2,3)
plot(0:length(y3)-1,y3)
u=rand(1,20);
x=(u>1/2)-(u<=1/2);
y4=[0 cumsum(x)];
subplot(2,2,4)
plot(0:length(y4)-1,y4)
```

Examine one of the realizations. Describe the motion of the drunkard based on this realization. (From one time instant to the next, the drunkard either takes one step forward or one step backward along a horizontal axis—start the drunkard at the origin.)

- *Example 8.* In this example you plot some realizations of the CT random sinusoid process

$$X(t) = A\sin(2\pi t + \Theta), \quad t \geq 0$$

where Θ, the "random phase", is uniformly distributed between 0 and 2π, and where A, the "random amplitude", is a standard gaussian RV. The plots are done only from $t = 0$ to $t = 3$ (three periods).

```
t=0:.01:3;
x1=randn(1,1)*sin(2*pi*t+2*pi*rand(1,1));
subplot(2,2,1)
plot(t,x1)
x2=randn(1,1)*sin(2*pi*t+2*pi*rand(1,1));
```

```
subplot(2,2,2)
plot(t,x2)
x3=randn(1,1)*sin(2*pi*t+2*pi*rand(1,1));
subplot(2,2,3)
plot(t,x3)
x4=randn(1,1)*sin(2*pi*t+2*pi*rand(1,1));
subplot(2,2,4)
plot(t,x4)
```

Is the amplitude of the realization more likely to be between 0 and 1 or more likely to be between 1 and 2? Generate a few more realizations until you feel you are ready to answer this question.

11.4 Exp 4: 1-D/2-D Cross-Sections of a Process

Let $X(t)$ be a random process associated with a random experiment. Fix any time t_0. Suppose you were to execute the following two steps:

Step 1: Perform the experiment and observe the realization signal $x(t)$ that you get.

Step 2: Sample $x(t)$ at time $t = t_0$, obtaining the value $x(t_0)$.

As the result of these two steps, you obtain the value of a *random variable* which we denote by $X(t_0)$. This RV $X(t_0)$ is called a "1-D cross-section" of the process $X(t)$. If you sample the process at different times, then you get different 1-D cross-sections.

Example 9. This example teaches you that 1-D cross-sections taken at different times can have quite a different statistical character. Perform the following steps:

(a) Run the lines of code:

```
t=0:.01:3;
a=2*floor(2*rand(1,2))-1;
x=a(1)*t+a(2);
plot(t,x)
```

You will see the plot of a realization of a process $X(t)$ on your screen.

(b) Run the lines of code in (a) 10 different times. On each run, look at the plot of the realization that you get and using your eyeball, sample the realization at time $t = 1$. Write down the sequence of 10 sample values that you get in your recitation notebook. These are 10 simulated values of the 1-D cross-section random variable $X(1)$.

(c) Repeat (b), now sampling each of your 10 realizations at time $t = 2$. The 10 sample values in your notebook now simulate 10 values of the 1-D cross-section random variable $X(2)$.

(d) Compare the results from (a) and (b) that you wrote down in your notebook. Do the cross-sections $X(1)$ and $X(2)$ seem to have different PMF's? Do you have any idea what these PMF's might be?

For a given random process $X(t)$, suppose you now fix two times t_0, t_1, with $t_0 < t_1$. Then, the pair of RV's $(X(t_0), X(t_1))$ is called a 2-D cross-section of process $X(t)$. You can observe a value of $(X(t_0), X(t_1))$ by (i) performing the underlying experiment and seeing what realization results, and then (ii) sampling that realization at times t_0, t_1, respectively. If these samples are a, b, respectively, then the point (a, b) is one observed value of the 2-D cross-section $(X(t_0), X(t_1))$. Performing the experiment repeatedly, you will get further observed values (a, b) of $(X(t_0), X(t_1))$.

Example 10. Run the script in part(a) of Example 9 ten times. Each time, sample the realization you see on your screen at times $t = 1$ and $t = 2$. This will give you 10 (a, b) points in your notebook, which are simulated values of the 2-D cross-section $(X(1), X(2))$. If you obtained thousands of (a, b) points in this way, you could average up the $a * b$ values to estimate the correlation $E[X(t_0)X(t_1)]$ between the process and time t_0 and time t_1. We will use this correlation estimation technique next week (where we will call this method "space-averaging" or "averaging across cross-sections"). If you have more time, estimate $E[X(1)X(2)]$ for the process of Examples 9-10 in this way.

11.5 Exp 5: Poisson Process Introduction

The Poisson process (also called Poisson arrival process) is used to model arrivals in a queueing system. In this experiment, we look at some realizations of a Poisson process and conclude various things from them.

Example 11. Let us first consider a Poisson process for which there is one arrival per second, on average. We can simulate the first six arrival times via the Matlab code:

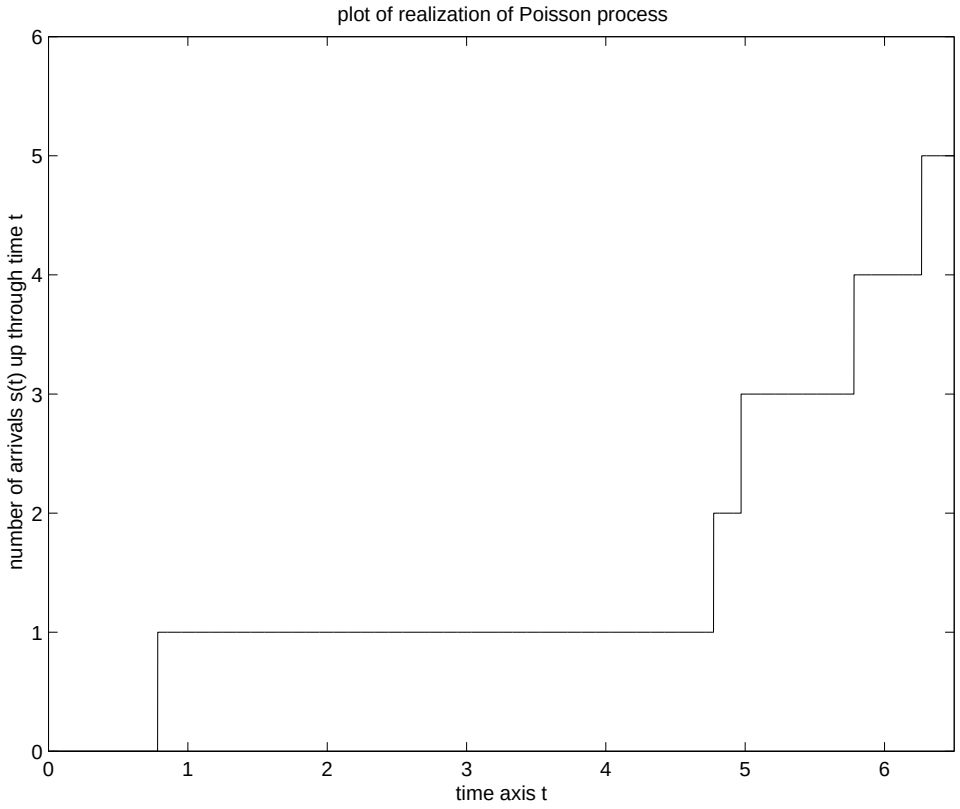
```
t=cumsum(-log(rand(1,6)));
```

Let these six arrival times be $t_1, t_2, t_3, t_4, t_5, t_6$; these are the entries of the vector `t`. Consider the step function defined $s(t)$ over the time interval $0 \leq t \leq t_6$ defined as follows: $s(t)$ is equal to 0 in the time interval $0 \leq t < t_1$, is equal to 1 in the time interval $t_1 \leq t < t_2$, 2 in the time interval $t_2 \leq t < t_3$, etc., ending up equal to 5 in the time interval $t_5 \leq t \leq t_6$. If we were to consider more and more arrivals until we had infinitely many, then the step function $s(t)$ would keep getting extended until the result would be a realization of the Poisson process. Plot $s(t)$ using Matlab, by running the following Matlab code:

```
t=cumsum(-log(rand(1,6)));
t=round(10^3*t)/10^3; %rounds arrival time to 3 decimal places
delta=.001;
x1=0:delta:t(1)-delta; y1=0*ones(1,length(x1));
x2=t(1):delta:t(2)-delta; y2=1*ones(1,length(x2));
x3=t(2):delta:t(3)-delta; y3=2*ones(1,length(x3));
x4=t(3):delta:t(4)-delta; y4=3*ones(1,length(x4));
```

```
x5=t(4):delta:t(5)-delta; y5=4*ones(1,length(x5));
x6=t(5):delta:t(6)-delta; y6=5*ones(1,length(x6));
plot([x1 x2 x3 x4 x5 x6],[y1 y2 y3 y4 y5 y6])
axis([0 t(6) 0 6])
xlabel('time axis t')
ylabel('number of arrivals s(t) up through time t')
title('plot of realization of Poisson process')
```

The resulting plot you see on your screen is a realization $s(t)$ of the Poisson process. Does it look a little bit like the plot above?



Look at your realization $s(t)$, and answer the following questions:

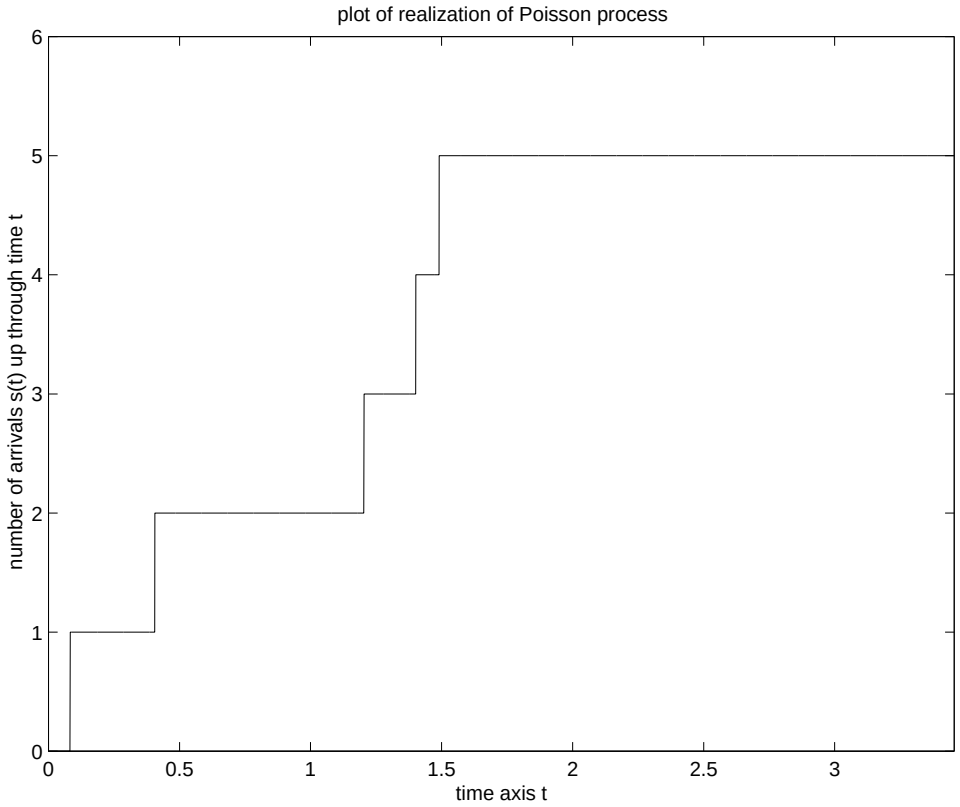
- At what times do each of the first 6 arrivals occur?
- What is the length of time between the first arrival and the second arrival, or between the second arrival and the fourth arrival?
- What does the arrival rate seem to be in number of arrivals per second (approximately)?

Re-run the preceding lines of code repeatedly to get other realizations. They should all give you roughly 6 arrivals in the first 6 seconds. But of course no two realizations will be the same.

Example 12. Suppose you now want the arrival rate of the Poisson process to be two arrivals per second. Then, you replace the first line of the Matlab script of Example 11 with

```
t=cumsum(-log(rand(1,6))/2);
```

Run the script of Example 11 after making this change and then look at the realization of the Poisson process that you see plotted on your Matlab screen. It should look a little bit like the following plot:



You can re-run the script to get still further realizations. On average, you will have 6 arrivals in 3 seconds, an arrival rate of 2 arrivals per second.

EE 3025 S2007 Recitation 11 Lab Form

Name and Student Number of Team Member 1:

Name and Student Number of Team Member 2:

Name and Student Number of Team Member 3:

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Study Experiment 1. You will be doing something in connection with the Experiment 1 framework.