6   Rec 6: Derived RV's, Joint Random Pairs (X, Y)

Directions: Your instructor will spend the first 40 minutes of the recitation period working some review problems and going over one or more Matlab experiments in the following. During the last 10 minutes of recitation, your proctor will give you a "Lab Form" that your recitation team completes, signs, and turns in. See the last page for an indication of what you will be asked to do on the Lab Form.

Due to time limitations, only a part of the following can be covered during the recitation period. However, you might want in the future to try some of the uncovered experiments on your own. They could give skills useful on some future homework problems and could lend insight into your understanding of the course from an experimental point of view.

During the last 10 minutes of recitation, your proctor will give you a "Lab Form" that your recitation team completes, signs, and turns in.

This Week's Topics.

- Derived RV's

- Joint Discrete (X, Y)

- Joint Uniform (X, Y)

- MAP Detector Design

- Introduction to Joint Gaussian (X, Y)

6.1   Exp 1: Derived RV's

The goal of this experiment is to give you a further understanding of Section 3.7 of the textbook on derived RV's.

Let X be a continuously dist'ed RV and let Y be a continuously dist'ed RV which is either a strictly increasing or strictly decreasing function of X. In class, I used the CDF method to obtain the density $f_Y(y)$ from the density $f_X(x)$. In this special case (because there is a one-to-one correspondence between the values of X and the values of Y), there is an easier method than the CDF method to find $f_Y(y)$. Namely, $f_Y(y)$ and $f_X(x)$ are related by the formula

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|. \tag{1}$$

Example 1. In this example, we present you with some convincing numerical evidence that the formula (1) is valid. Suppose Y and X are related by the formula

$$Y = \sqrt{X}.$$

(For this to make sense, we suppose that X just takes nonnegative values, and that the square root is the positive square root.) Then, formula (1) becomes

$$f_Y(y) = 2y f_X(y^2).\qquad(2)$$

At this point, you should take out your notebook and show that formula (2) does indeed follow from formula (1). Now suppose that the RV X is Uniform(0,1). Then the density $f_Y(y)$ of $Y = \sqrt{X}$ is given by

$$f_Y(y) = \begin{cases} 2y, & 0 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

Notice that this is a ramp function with slope 2 that goes from $y = 0$ to $y = 1$. In the following, you will convince yourself that this is true by simulating values of Y and using these simulated values to estimate the density $f_Y(y)$ as you did in Recitation 3. Perform the following steps:

Step 1: Generate 100000 pseudorandom observations of the random variable Y by running the Matlab script:

```
clear;
n=100000;
x=rand(1,100000);
y=sqrt(x);
```

Step 2: Run the following Matlab script, which compares an estimate of the plot of $f_Y(y)$ (based on the 100000 Y samples) to the actual plot of $f_Y(y)$:

```
N=n/100;
a=min(y); b=max(y);
Delta=(b-a)/N;
t=a-Delta/2+[1:N]*Delta;
f=hist(y,t)/(Delta*n);
subplot(2,1,1) %plot estimated density
bar(t,f)
PDF=2*t;
subplot(2,1,2) %plot actual density
plot(t,PDF)
```

The top plot in your figure on the screen is the estimated density and the bottom one is the actual density for Y. Do you get pretty close agreement?

Step 3: To make the preceding more convincing, you can plot the estimated CDF of Y versus the actual CDF of Y. Before running the following code, close the plotting window left over from Step 2.

```
p=hist(y,t)/n;
CDFestimated=cumsum(p); %estimated CDF from Y data
CDFactual=t.^2;  %actual CDF computed from density of Y
plot(t,CDFestimated,t,CDFactual) %two plots plotted top of one another
```

Does your plot look like just one curve?

Example 2. In one of your homework problems for March 4, you are asked to investigate a change of variable using the Matlab function "`erfinv`". Specifically, suppose we define a RV

$$Y = \text{erfinv}(U),$$

where U is Uniform(0,1). Run the following Matlab script, which will give an estimated density for Y based upon 100000 simulated observations of the value of Y:

```
clear
n=100000;
y=erfinv(rand(1,n));
N=n/100;
a=min(y); b=max(y);
Delta=(b-a)/N;
t=a-Delta/2+[1:N]*Delta;
f=hist(y,t)/(Delta*n);
bar(t,f)
```

Does your estimated density plot look like one half of a Gaussian bell-shaped curve? How do you think you can modify the above Matlab code to obtain both halves of the Gaussian bell-shaped curve? Hint: Instead of the line

```
y=erfinv(rand(1,n));
```

try a line of the form

```
y=erfinv(A*rand(1,n)+B);
```

where the constants A and B are adjusted to give you uniform data over some other interval than $[0,1]$; your interval should be a symmetric interval of the form $[-c,c]$ for some $c > 0$. Once you figure out what to take as A and B above, you will be well on your way towards solving the Homework Problem involving "erfinv".

6.2   Exp 2: Joint Discrete (X, Y)

In Example 16.2 of the class notes, I introduced an "ice cream cone experiment" which resulted in a joint pair of discrete RV's (X, Y). In this Matlab experiment, you will carry Example 16.2 a little bit further. First, let me refresh your memory about this experiment. First, Bill eats X ice cream cones, where X is a Poisson RV with mean 1. Then, Bill flips

3

a fair coin $X + 1$ times, observes the number of heads Y, and then runs this many miles in order to work off the effects of his ice-cream binge. The values of (X, Y) are the set of points

$$S = \{(x, y) : x = 0, 1, 2, 3, \cdots; \ y = 0, 1, \cdots, x + 1\}.$$

In Example 16.2, we derived the joint PMF $P^{X,Y}(x, y)$ to be as follows:

$$P^{X,Y}(x, y) = \begin{cases} (\exp(-1)/x!)\binom{x+1}{y}(1/2)^{x+1}, & (x, y) \in S \\ 0, & (x, y) \notin S \end{cases}$$

In the following Examples, you use Matlab to determine various things about this joint PMF.

Example 3. It should be true that

$$\sum_{x=0}^{\infty} \sum_{y=0}^{x+1} P^{X,Y}(x, y) = 1.$$

Write some Matlab code using two nested for loops (one for x and one for y) to obtain the sum

$$\sum_{x=0}^{N} \sum_{y=0}^{x+1} p_{X,Y}(x, y) \tag{3}$$

for a large N (try $N = 10, 25, 50$). You can use the fact that the Matlab syntax for $P^{X,Y}(x, y)$ is:

```
exp(-1)*(1/2)^(x+1)*(x+1)/(prod(1:y)*prod(1:x+1-y))
```

If your program crashes, your instructor will try to help you. For each N you try, see how close to 1 your sum (3) is. Use "`format long`" to see if you are getting accuracy beyond the fourth decimal place.

Example 4. The RV Y (the number of miles Bill runs) takes the values $0, 1, 2, \cdots$. Its marginal PMF $P^Y(y)$ is given by the formulas:

$$P^Y(0) \ = \ P(Y = 0) = \sum_{x=0}^{\infty} P^{X,Y}(x, 0)$$

$$P^Y(y) \ = \ P(Y = y) = \sum_{x=y-1}^{\infty} P^{X,Y}(x, y), \ \ y = 1, 2, \cdots \tag{4}$$

In particular, we have

$$P(Y = 0) = \sum_{x=0}^{\infty} (\exp(-1)/x!)(1/2)^{x+1}. \tag{5}$$

Write some Matlab code and run it to get a good approximation to $P(Y = 0)$. (Hint: You could again use the above Matlab syntax for $P^{X,Y}(x, y)$ where you use a double for loop in which one of the loops is "`for y=0:0`".) Now go back to equation (5) and sum up the series on the right side using the following McClaurin series expansion:

$$e^t = 1 + t + t^2/2 + t^3/6 + t^4/24 + \cdots. \tag{6}$$

4

You should be able to prove that

$$P(Y = 0) = \exp(-0.5)/2.$$

Compare this exact value with the approximation for $P(Y = 0)$ you got earlier. Now run the following Matlab script, which simulates 5000 observations of X and of Y:

```
clear
for i=1:5000
N=-1;
T=0;
while T<1
T=T-log(rand(1,1));
N=N+1;
end
x(i)=N;
y(i)=sum(rand(1,N+1)>1/2);
end
```

Execute the line

```
mean(y==0)
```

Do you get an approximation to $P(Y = 0)$ that is good to two decimal places?

Example 5. The following formula follows from (4):

$$P(Y = 1) = \sum_{x=0}^{\infty} (\exp(-1)/x!)(x + 1)(1/2)^{x+1}.$$

Write some Matlab code and run it to get a good approximation to $P(Y = 1)$. You can do it two ways: (1) with the preceding formula, and (2) with a simulation as at the end of Example 4.

*Example 6.* The following formula follows from (4):

$$P(Y = 2) = \sum_{x=1}^{\infty} (\exp(-1)/x!)(x + 1)(x)(1/2)^{x+2}.$$

Use Matlab to get an approximation to $P(Y = 2)$ two different ways.

Final Remark. Using the McClaurin Series formula (6), one can obtain a closed form expression for $P^Y(y)$ that is valid for every nonnegative integer y. (You worked out the exact value for $P^Y(0)$ in Example 4. The exact derivations of $P^Y(y)$ for $y > 0$ are not that much harder.) When one does this, one sees that Y is not Poisson, even though X is Poisson. (Y turns out to have a PMF that is not one of our common distributions from Chapters 2-3.)

6.3   Exp 3: Joint Uniform (X, Y)

Let R be the following triangular region in the (x, y)-plane:
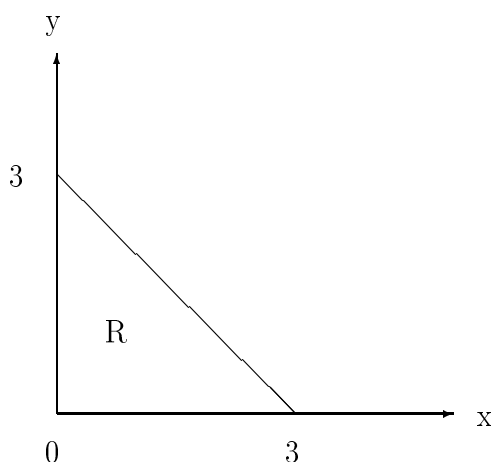
Suppose we choose the random pair (X, Y) uniformly from R. From the class notes, we know that the joint density $f_{X,Y}(x, y)$ is constant over R, with the constant being the reciprocal of the area of R. The reader can now verify that this gives us the following expression for the joint density:

$$f_{X,Y}(x, y) = \begin{cases} 2/9, & (x, y) \in R \\ 0, & \text{elsewhere} \end{cases}$$

Example 7. In this example, I am going to show you how to simulate observations of (X, Y) and how to use these simulated observations to estimate a probability involving (X, Y). Run the following program, which creates a vector x and a vector y consisting of 5000 simulated observations of X and Y, respectively:

```
clear;
i=0;
while i<5000
x_temp=3*rand(1,1);
y_temp=3*rand(1,1);
if x_temp+y_temp<3
i=i+1;
x(i)=x_temp; y(i)=y_temp;
else
end
end
```

(The program works by randomly generating a point (x, y) in a 3×3 square, and then keeping the point (x, y) as a simulated value of (X, Y) if it lies in R.) Now we are going to use these simulated observations of X and Y to estimate the probability

$$P[X^2 + Y^2 < 4]. \tag{7}$$

Run the following line of Matlab code in order to obtain the estimate:

6

```
mean(x.^2+y.^2<4)
```

The precise answer is the area of the circular quadrant

$$\{(x, y) : x^2 + y^2 < 4, \ x \geq 0, \ y \geq 0\}$$

divided by the area of R. Using this fact, compute the exact value of (7). (Recall that the area of a circle is π times the square of the radius.) Compare the exact value of (7) with the estimate you obtained above. Do you get agreement to two or three decimal places?

Example 8. We find the marginal density $f_X(x)$ of X by doing the following:

$$f_X(x) = \int_0^{3-x} (2/9)dx = (2/9)(3 - x), \ 0 \leq x \leq 3$$

(The PDF $f_X(x)$ is zero elsewhere.) Verify that this is a density by running the Matlab script:

```
clear;
syms x
C=2/9;
int(C*(3-x),0,3)
```

(The value of the integral should be 1.) Now compute the mean $E[X]$ by running the Matlab script:

```
clear;
syms x
C=2/9;
int(C*x*(3-x),0,3)
```

Now re-run the script from Example 8 which will generate vectors **x** and **y** simulating 5000 observations of X and Y, respectively. Then, execute the command

```
mean(x)
```

and see how close this estimate is to $E[X]$.

Example 9. The area inside an ellipse is πab, where a is the major radius and b is the minor radius. In this example, you are going to estimate the area inside the ellipse

$$x^2/9 + y^2/4 = 1, \tag{8}$$

which has major radius $a = 3$ and minor radius $b = 2$. First, run the script:

```
A=-3; B=3;
x=(B-A)*rand(1,50000)+A;
y=(B-A)*rand(1,50000)+A;
```

Suppose you were to select a random point (X, Y) uniformly from the square

$$\{(x, y) : -3 \leq x \leq 3; \; -3 \leq y \leq 3\}.$$

What you have stored in Matlab memory are vectors **x** and **y** that simulate 50000 observations of X,Y, respectively. Now execute a line of Matlab code which will use **x** and **y** to obtain an estimate of

$$P[X^2/9 + Y^2/4 \leq 1]. \tag{9}$$

(Use Example 7 as a guide for doing this.) Now use your estimate of (9) to obtain an estimate of the area inside the ellipse (8). Compare this estimate to the exact area obtained by using the formula πab.

6.4 Exp 4: MAP Detector Design

For this experiment, you need to understand Section 17.2 of the Class Notes. Better take a look at that now if you have not yet done so. (You can access it from your current location.) Otherwise, what I'm attempting to do in the following example will be totally meaningless to you. (If you would rather take the time you need to understand the MAP detector outside of recitation, you can skip ahead at this point to Experiment 5.)

Example 10. The channel matrix of a discrete channel model is given to be

$$
\begin{array}{c}
\\
X = 0 \\
X = 1 \\
X = 2
\end{array}
\begin{array}{ccc}
Y = 0 & Y = 1 & Y = 2 \\
\left(
\begin{array}{ccc}
1/2 & 1/2 & 0 \\
0 & 1/2 & 1/2 \\
1/2 & 0 & 1/2
\end{array}
\right)
\end{array}
\tag{10}
$$

The channel input probabilities are

$$P(X = 0) = 0.32, \quad P(X = 1) = 0.33, \quad P(X = 2) = 0.35.$$

Use Matlab to obtain the joint PMF matrix for the channel input-output pair (X, Y). Let us denote the MAP detector output as $\hat{X}$, as in the following block diagram:

$$X \rightarrow \boxed{\text{channel}} \rightarrow Y \rightarrow \boxed{\text{detector}} \rightarrow \hat{X}$$

By staring at the joint PMF matrix on your screen, determine the three MAP detector outputs that go in the question marks below:

$$
\begin{aligned}
Y = 0 &\implies \hat{X} = ? \\
Y = 1 &\implies \hat{X} = ? \\
Y = 2 &\implies \hat{X} = ?
\end{aligned}
$$

Choose the largest probability from each column of your joint PMF matrix. Add up these three probabilities and subtract from 1. What you have computed is $P[X \neq \hat{X}]$, the probability that the MAP detector does not estimate X correctly. Keep a record of this $P[X \neq \hat{X}]$ value that you have just computed; you are going to estimate it in the next example.

Example 11. Run the following Matlab script, which simulates 50000 inputs and outputs for the discrete channel model in Example 10:

```
clear
u=rand(1,50000);
x=0*(u<0.32)+1*(u>=0.32 & u<0.65)+2*(u>=0.65);
z=floor(2*rand(1,50000));
y=rem(x+z,3);
```

Let a, b, c be the MAP detector outputs for respective inputs $y = 0, 1, 2$ that you found in Example 10. Run the following Matlab one-liner:

```
xhat=a*(y==0)+b*(y==1)+c*(y==2);
```

What you have done is to store 50000 MAP detector outputs. Now execute the line of Matlab code:

```
mean(x~=xhat);
```

What this furnishes you is an estimate of the MAP detector error probability $P[X \neq \hat{X}]$ that you computed in Example 10. Compare the estimate you just got to the actual value of $P[X \neq \hat{X}]$.

6.5 Exp 5: Introduction to Joint Gaussian (X, Y)

A random pair (X, Y) is *joint Gaussian* (or *bivariate Gaussian*) if its PDF is as given on page 191 of your textbook. In this experiment, I will look at a simplified form of this joint density, which is

$$f(x, y) = \frac{1}{2\pi\sqrt{1 - \rho^2}} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1 - \rho^2)}\right) \tag{11}$$

The parameter ρ is strictly between -1 and 1 and is called *correlation coefficient*. We will learn more about the correlation coefficient in subsequent classes. In this experiment, I am going to show you what happens to the plot of the surface

$$z = f(x, y)$$

as you change ρ. This will give you a little bit of an insight into what ρ might mean. Specifically, you will learn the following:

- For $\rho = 0$, cross-sections of the surface $z = f(x, y)$ with planes parallel to xy-plane are circles (Example 12).

- For $\rho \neq 0$, cross-sections of the surface $z = f(x, y)$ with planes parallel to xy-plane are rotated ellipses (Example 13).

- As ρ approaches ± 1 the surface $z = f(x, y)$ gets squashed along a straight line (Example 14-15).

Example 12. Run the Matlab script:

```
clear;
x=-4:.2:4; %creates  x-axis for surface plot
y=-4:.2:4; %creates  y-axis for surface plot
[X,Y]=meshgrid(x,y); %creates points in xy-plane
rho=0;
Z=1/(2*pi*sqrt(1-rho^2))*exp(-(X.^2-2*rho*X.*Y + Y.^2)/(2*(1-rho^2)));
colormap(gray)
mesh(X,Y,Z)
```

[X,Y] denotes a grid of points in the xy-plane. Z, stored in Matlab memory, is a matrix giving the height z=f(x,y) of the density surface above each point (x,y) in the rectangular grid [X,Y]. On your screen, you see the plot of the density surface. The plot is for the special case in which $\rho = 0$. In this case, the joint Gaussian density is

$$\frac{1}{2\pi} \exp(-(x^2 + y^2)/2),$$

which is the case of two independent standard Gaussian RV's X, Y. In the surface plot, you will see the curves of intersection of the surface with planes of form x=C and y=C. Do these look like the bell-shaped Gaussian density curves? Execute the following commands one by one and see what effect they each have on your surface plot:

```
surfl(X,Y,Z)
surfl(X,Y,Z), shading interp
```

Execute the command:

```
contour(X,Y,Z)
```

These are the cross-sections of the surface plot with planes parallel to the xy-plane. Do these look like ellipses? Actually, they are supposed to be circles (they might not look like circles on the plot because the x and y axes may not be to same scale). To get more of the cross-sections, execute the line

```
contour(X,Y,Z,40)
```

  Example 13. Execute the lines

```
clear;
x=-4:.2:4;
y=-4:.2:4;
[X,Y]=meshgrid(x,y);
rho=1/2;
Z=1/(2*pi*sqrt(1-rho^2))*exp(-(X.^2-2*rho*X.*Y + Y.^2)/(2*(1-rho^2)));
colormap(gray)
mesh(X,Y,Z)
```

Now execute the following lines one by one and examine each of the plots you get:

```
surfl(X,Y,Z), shading interp
contour(X,Y,Z)
```

Do the ellipses in your cross-section seem rotated? Find the equations of these ellipses from the exponent of the density in (11). Do the equations contain an xy term? (It is the xy term in the equation of an ellipse that makes it rotated.)

Example 14. Execute the lines:

```
x=-4:.05:4;
y=-4:.05:4;
[X,Y]=meshgrid(x,y);
rho=0.95;
Z=1/(2*pi*sqrt(1-rho^2))*exp(-(X.^2-2*rho*X.*Y + Y.^2)/(2*(1-rho^2)));
colormap(gray)
mesh(X,Y,Z)
surfl(X,Y,Z), shading interp
```

Does the surface plot appear squashed along a straight line in the xy-plane?

Example 15. Execute the lines:

```
x=-4:.05:4;
y=-4:.05:4;
[X,Y]=meshgrid(x,y);
rho=-0.95;
Z=1/(2*pi*sqrt(1-rho^2))*exp(-(X.^2-2*rho*X.*Y + Y.^2)/(2*(1-rho^2)));
colormap(gray)
mesh(X,Y,Z)
surfl(X,Y,Z), shading interp
```

Does the surface plot appear squashed along a straight line in the xy-plane?

Remark. We will prove in class that if a random pair (X, Y) has correlation coefficient $\rho = \pm 1$, then there is a straight line relationship between X and Y, that is, there is a straight line in the xy-plane such that every observed value of (X, Y) will fall on this straight line. This explains why the joint Gaussian density plot $z = f(x, y)$ got more squashed along a straight line as $\rho \to \pm 1$.

In the future, you will learn more about the joint Gaussian distribution that what you have learned today.

EE 3025 S2007 Recitation 6 Lab Form

Name and Student Number of Team Member 1:

Name and Student Number of Team Member 2:

Name and Student Number of Team Member 3:

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Study Experiment 3. In this week's lab reports, you will estimate the area or volume of a
2-D or 3-D geometrical figure via a Matlab simulation.