EE 3025 S2005 Final Exam Review Questions

1. Airline A and Airline B are the only two airlines serving the city of Metropolis. An Airline A outgoing flight has a late departure with probability 0.25, whereas an Airline B outgoing flight has a late departure with probability 0.40. On each of Lois Lane's trips out of Metropolis, she flips her lucky coin; if the coin comes up heads, she takes Airline A, and if the coin comes up tails, she takes Airline B.

(a) After traveling for many years, Lois concludes that she has a late departure from Metropolis with probability 1/3. Is her lucky coin a fair coin? What is the probability that her lucky coin will come up heads?

Solution. The array of "forward conditional probabilities" is

$$\begin{array}{c} \\ A \\ B \end{array} \begin{array}{cc} \textit{on time} & \textit{late} \\ \end{array}$$
$$\begin{array}{c} A \\ B \end{array} \left(\begin{array}{cc} .75 & .25 \\ .6 & .4 \end{array} \right)$$

$$1/3 = P(late) = [P(A)\ \ P(B)] \cdot [.25\ \ 4] = .25P(H) + .4(1 - P(H)).$$

Solving, we get

$$P(H) = 4/9.$$

The coin is not fair.

(b) Given that Lois's flight leaves Metropolis on time, what is the probability she flew on Airline B?

Solution.

$$P(B|on\ time) = \frac{P(B \cap \{on\ time\})}{P(on\ time)} = \frac{P(B)P(on\ time|B)}{2/3} = \frac{(5/9)(0.6)}{2/3} = 1/2.$$

2. Use the table on page 123 of your textbook to answer the following questions concerning Gaussian RV's.

(a) A Gaussian RV X satisfies

$$\begin{aligned} P(X \leq 5.65) &= 0.7939 \\ P(X \leq 7.4) &= 0.9357 \end{aligned}$$

Compute the mean μ and the standard deviation σ of X.

Solution. Let Z be the standard Gaussian RV

$$Z = \frac{X - \mu}{\sigma}.$$

From the table, since

$$P(Z \leq \frac{5.65 - \mu}{\sigma}) = 0.7939,$$

we must have

$$\frac{5.65 - \mu}{\sigma} = 0.82 \qquad (1)$$

Also, since

$$P(Z \leq \frac{7.4-\mu}{\sigma}) = 0.9357,$$

we must have

$$\frac{7.4-\mu}{\sigma} = 1.52 \qquad (2)$$

Solving equations (1)-(2) simultaneously, you'd be able to compute the values of μ and σ.

(b) A random sample of size 100 is taken from a Gaussian distribution with mean 0 and standard deviation 6. Let $\bar{X}$ be the sample mean of this random sample. Compute $P(-0.5 \leq \bar{X} \leq 0.5)$.

Solution. $\bar{X}$ is Gaussian with mean 0 and standard deviation $6/\sqrt{100} = 0.6$. Therefore,

$$Z = \frac{\bar{X}}{.6}$$

is standard Gaussian.

$$P(-0.5 \leq \bar{X} \leq 0.5) = P(-5/6 \leq Z \leq 5/6) = 2\Phi(5/6) - 1 = 0.596.$$

3. 6 coins are tossed. Coins 1,2,3 are fair, but coins 4,5,6 are unfair, each satisfying $P(H) = 2/3$. Let X be the number of heads you obtain among the three fair coins. Let Y be the number of heads you obtain among the three unfair coins. Using the binomial RV's X, Y, answer the following questions.

(a) Compute $P(\text{all 6 coins are heads})$.

Solution.

$$P(\text{all 6 coins are heads}) = P(X=3)P(Y=3) = [(1/2)^3] * [(2/3)^3].$$

(b) Compute $P(\text{at least one head})$.

Solution. This is the same thing as 1 minus the prob of no heads.

$$1 - P(\textit{no heads}) = 1 - P(X=0)P(Y=0) = 1 - [(1/2)^3] * [(1/3)^3].$$

(c) Compute $P(\text{exactly 3 heads})$.

Solution. This is

$$P(X=0)P(Y=3)+P(X=1)P(Y=2)+P(X=2)P(Y=1)+P(X=3)P(Y=0),$$

which reduces to

$$[(1/2)^3]*[(2/3)^3]+[3(1/2)^3]*[3(2/3)^2(1/3)]+[3(1/2)^3]*[3(2/3)(1/3)^2]+[(1/2)^3]*[(1/3)^3].$$

4. Let $X(t)$ be WSS with $R_X(\tau) = \exp(-|\tau|)$ be passed through a filter with $h(t) = \exp(-2t)u(t)$. Compute power P_Y generated by the filter output process $Y(t)$.

Solution. The frequency response of the filter is

$$H(f) = \frac{1}{j2\pi f + 2}$$

and therefore

$$|H(f)|^2 = \frac{1}{(2\pi f)^2 + 4}$$

We also have

$$S_X(f) = \frac{2}{(2\pi f)^2 + 1}$$

The output power is therefore

$$\begin{aligned} P_Y &= \int_{-\infty}^{\infty} S_Y(f)df \\ &= \int_{-\infty}^{\infty} S_X(f)|H(f)|^2 df \\ &= \int_{-\infty}^{\infty} \frac{2}{((2\pi f)^2 + 1)((2\pi f)^2 + 4)} df \\ &= \int_{-\infty}^{\infty} \left[\frac{2/3}{(2\pi f)^2 + 1} + \frac{-2/3}{(2\pi f)^2 + 4}\right] df \\ &= [1/3 - 1/6] = 1/6 \end{aligned}$$

5. Let $X(t)$ and $Y(t)$ be independent WSS peocesses with the following power spectral densities:

$$S_X(f) = \begin{cases} 3, & -5 \le f \le 5 \\ 0, & elsewhere \end{cases}$$

$$S_Y(f) = \begin{cases} 4, & -5 \le f \le 5 \\ 0, & elsewhere \end{cases}$$

Let $Q(t)$ be the random signal

$$Q(t) = X(t)Y(t).$$

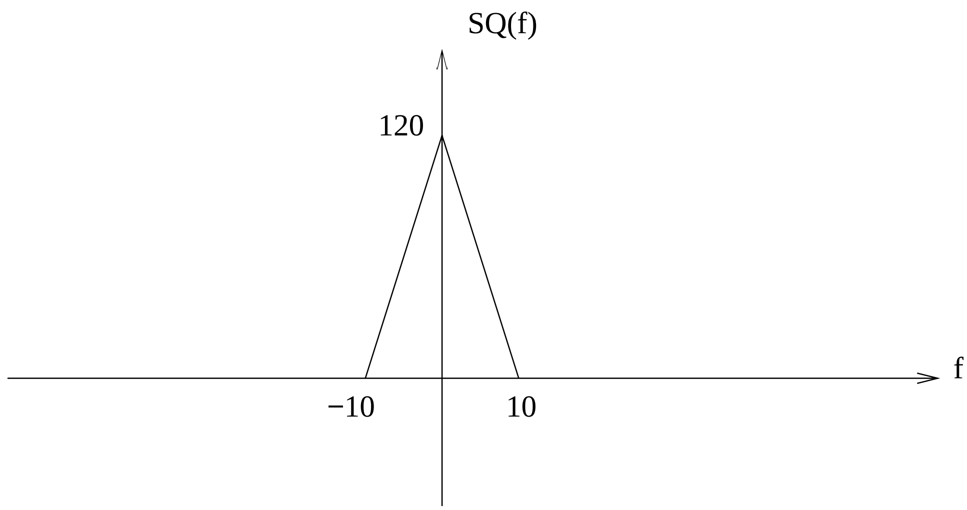
Find $S_Q(f)$ and P_Q.

Solution. We know from earlier work that

$$R_Q(\tau) = R_X(\tau)R_Y(\tau). \tag{3}$$

Plugging in $\tau = 0$, we get

$$P_Q = R_Q(0) = R_X(0)R_Y(0) = P_X P_Y = \int S_X(f)df \int S_Y(f)df = 30 * 40 = 1200.$$



Fourier transforming equation (3), we obtain

$$S_Q(f) = S_X(f) * S_Y(f).$$

We know from EE 3015 that $S_Q(f)$ automatically will be a symmetric triangular pulse extending from $f = -10$ to $f = 10$. The height of the triangle must be 120 in order for the area under it to be 1200. So the plot of $S_Q(f)$ is the the plot given above.

6. Let X_n ("the signal") and Z_n ("the noise") be WSS processes satisfying

$$R_X(\tau) = \begin{cases} 1, & \tau = 0 \\ 1/2, & \tau = \pm 1 \\ 0, & elsewhere \end{cases}$$

$$R_Z(\tau) = \begin{cases} 1, & \tau = 0 \\ 0, & elsewhere \end{cases}$$

We filter as follows:

$$X_n + Z_n \rightarrow \boxed{h[n]} \rightarrow X_n^0 + Z_n^0$$

where the filter impulse response is

$$h[n] = \begin{cases} 1, & n = 0, 1 \\ 0, & elsewhere \end{cases}$$

(X_n^0 and Z_n^0 are the filter responses to X_n and Z_n, respectively.)

(a) Compute the filter input SNR.

Solution.

$$filter\ input\ SNR = \frac{P_X}{P_Z} = \frac{R_X(0)}{R_Z(0)} = 1.$$

(b) Compute the filter output SNR.

Solution.

$$filter\ output\ SNR = \frac{E[(X_n^0)^2]}{E[(Z_n^0)^2]}.$$

By Theorem 5.13 of your textbook,

$$\begin{aligned} E[(X_n^0)^2] &= \begin{bmatrix} h[0] & h[1] \end{bmatrix} \begin{bmatrix} R_X(0) & R_X(1) \\ R_X(1) & R_X(0) \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \end{aligned}$$

$E[Z_n^0)^2]$ is a similar calculation, with the "matrix in the middle" equal to

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

You get

$$E[(Z_n^0)^2] = [1\ \ 1] \cdot [1\ \ 1] = 2.$$

The output SNR is therefore 3/2.

7. WSS process X_n is filtered as follows:

$$X_n \to \boxed{H(z)} \to Y_n$$

The filter transfer function is

$$H(z) = \frac{2 + z^{-1}}{1 + z^{-1} + 4z^{-2}}.$$

(a) Compute $S_Y(f)$ assuming that X_n is white noise with unit variance.

Solution. We have

$$S_X(f) = 1,$$

and

$$H(z)H(z^{-1}) = \frac{5 + 2(z + z^{-1})}{18 + 5(z + z^{-1}) + 4(z^2 + z^{-2})}.$$

Substituting in $z = \exp(j2\pi f)$, we obtain

$$S_Y(f) = S_X(f)H(f)H(-f) = \frac{5 + 4\cos(2\pi f)}{18 + 10\cos(2\pi f) + 8\cos(4\pi f)}.$$

(b) What is μ_Y if $\mu_X = 4$?

Solution. We have

$$\mu_Y = \mu_X H(z)_{z=1} = 4 * (3/6) = 2.$$

8. A continuous-time process X is WSS and ergodic. One of its realizations $x(t)$ is periodic with period 6 and is defined over one period (extending from $t = -3$ to $t = 3$) as follows:

$$x(t) = \begin{cases} 0, & -3 \le t < -2 \\ 6, & -2 \le t \le 2 \\ 0, & 2 < t \le 3 \end{cases}$$

(All the other realizations are either forward or backward translations of $x(t)$.)

(a) Compute the process mean μ_X via the time-averaging technique.

Solution.

$$\mu_X = \frac{1}{6}\int_{-3}^{3} x(t)dt = 4.$$

(b) Compute $R_X(0)$ via the time-averaging technique.

Solution.

$$R_X(0) = \frac{1}{6}\int_{-3}^{3} x(t)^2 dt = 24.$$

(c) Compute $R_X(2)$ via the time-averaging technique.

Solution.

$$R_X(2) = \frac{1}{6}\int_{-3}^{3} x(t)x(t-2)dt = 12.$$

(d) Compute $Var[X(52) + X(50)]$.

Solution.

$$\begin{aligned} Var[X(52) + X(50)] &= Var[X(52)] + Var[X(50)] + 2Cov(X(52), X(50)) \\ &= 2(24 - 16) + 2 * (12 - 16) = 8. \end{aligned}$$

9. A discrete-time process X is WSS but nonergodic. It has exactly four realizations $x_1[n]$, $x_2[n]$, $x_3[n]$, $x_4[n]$ (occurring with prob 0.25 each) defined as follows:

$$\begin{aligned} &x_1[n] = 1 \text{ for even } n \text{ and } x_1[n] = 2 \text{ for odd } n \\ &x_2[n] = 2 \text{ for even } n \text{ and } x_2[n] = 1 \text{ for odd } n \\ &x_3[n] = 3 \text{ for even } n \text{ and } x_3[n] = 6 \text{ for odd } n \\ &x_4[n] = 6 \text{ for even } n \text{ and } x_4[n] = 3 \text{ for odd } n \end{aligned}$$

(a) Compute the process mean μ_X via the space-averaging technique.

Solution.

$$\mu_X = \frac{x_1[n] + x_2[n] + x_3[n] + x_4[n]}{4} = 3.$$

(b) Compute the process variance σ_X^2 via the space-averaging technique.

Solution.

$$R_X(0) = E[X[n]^2] = \frac{x_1[n]^2 + x_2[n]^2 + x_3[n]^2 + x_4[n]^2}{4} = 12.5.$$

$$\sigma_X^2 = R_X(0) - \mu_X^2 = 3.5.$$

(c) Compute $R_X(1)$ via the space-averaging technique.

Solution.

$$R_X(1) = \frac{x_1[n]x_1[n+1] + x_2[n]x_2[n+1] + x_3[n]x_3[n+1] + x_4[n]x_4[n+1]}{4} = 10.$$

10. A continuous-time WSS process X has autocorrelation function

$$R_X(\tau) = 20\cos(\pi|\tau|) + 100\exp(-0.5|\tau|).$$

(a) A first order predictor

$$\hat{X}(t+2.5) = AX(t)$$

is to be designed, in which the constant A is chosen to minimize the mean square prediction error $E[(X(t+2.5) - \hat{X}(t+2.5))^2]$. Using the orthogonality principle, compute the value of A.

Solution. The prediction error $X(t+2.5) - \hat{X}(t+2.5)$ must be orthogonal to $X(t)$. This tells us that

$$E[X(t)X(t+2.5)] = E[X(t)\hat{X}(t+2.5)],$$

which reduces to

$$R_X(2.5) = AR_X(0).$$

It follows that

$$A = R_X(2.5)/R_X(0) = (5/6)e^{-1.25}.$$

(b) A second order predictor

$$\hat{X}(t+2.5) = BX(t) + CX(t-1.5)$$

is to be designed, in which the constants B, C are chosen to minimize the prediction error $E[(X(t+2.5) - \hat{X}(t+2.5))^2]$. The system of equations you solve to obtain B, C takes the matrix form

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} B \\ C \end{bmatrix} = \begin{bmatrix} a_5 \\ a_6 \end{bmatrix},$$

where each a_i is a certain constant. Compute the values of the 6 constants $a_1, a_2, a_3, a_4, a_5, a_6$. DO NOT SOLVE FOR B and C.

Solution. The orthogonality relations tell us that

$$\begin{aligned} E[X(t)\hat{X}(t+2.5)] &= E[X(t)X(t+2.5)] \\ E[X(t-1.5)\hat{X}(t+2.5)] &= E[X(t-1.5)X(t+2.5)] \end{aligned}$$

These simplify to

$$\begin{aligned} BR_X(0) + CR_X(1.5) &= R_X(2.5) \\ BR_X(1.5) + CR_X(0) &= R_X(4) \end{aligned}$$

11. A continuous-time process X has power spectral density

$$S_X(f) = \cos\left(\frac{\pi f}{60}\right), \quad -30 \leq f \leq 30,$$

and equal to zero for all other f.

(a) Compute the power generated by the process X.

Solution.

$$P_X = \int_{-30}^{30} \cos(\pi f/60) df = 120/\pi.$$

(b) Process X is passed through an ideal low-pass filter with frequency response function

$$H(f) = 1, \quad -B \leq f \leq B,$$

and equal to zero for all other f. Compute the value of the bandwidth B (correct to one decimal place) so that the filter output process will generate exactly 95% of the power generated by X.

Solution. The filter output power is

$$\int_{-B}^{B} \cos(\pi f/60) df = \frac{120 \sin(B\pi/60)}{\pi}.$$

Setting this equal to 0.95 times $120/\pi$ leads us to the equation

$$\sin(B\pi/60) = 0.95,$$

from which we see that

$$B = \frac{60 * \mathrm{Sin}^{-1}(0.95)}{\pi}.$$

12. In each part of this problem, the discrete-time linear time-invariant filter with impulse response

$$h[n] = 2\delta[n] - 3\delta[n-1] + 4\delta[n-2]$$

is used to filter a random input signal.

(a) The input X to the filter is the discrete-time white noise process with zero mean and unit variance. The output process Y has power spectral density of the form

$$S_Y(f) = A + B\cos(2\pi f) + C\cos(4\pi f).$$

Compute the constants A, B, C.

Solution. First, determine that

$$h[n] * h[-n] = 29\delta[n] - 18(\delta[n-1] + \delta[n+1]) + 8(\delta[n-2] + \delta[n+2]). \quad (4)$$

It follows easily from this that

$$A = 29, \quad B = -36, \quad C = 16.$$

(b) The input X to the filter is a discrete-time WSS process with mean $\mu_X = 7$. Compute the mean μ_Y of the filter output process Y.

Solution.

$$\mu_Y = \mu_X[\sum h[n]] = 7 * 3 = 21.$$

(c) The input X to the filter is a discrete-time WSS process with

$$R_X(\tau) = 8\delta[\tau] + 4(\delta[\tau - 1] + \delta[\tau + 1]).$$

Compute the power generated by the filter output process.

Solution. The output power is the coefficient of the $\delta[n]$ component of $R_X(n)$ convoluted with $h[n] * h[-n]$. Use superposition: (1) convoluting $\delta[n]$ with $h[n] * h[-n]$ gives $\delta[n]$ coefficient of 29, by (4); (2) convoluting $\delta[n-1]$ with $h[n]*h[-n]$ gives coeff of $\delta[n]$ equal to -18, the coeff of $\delta[n+1]$ in $h[n] * h[-n]$; and (3) convoluting $\delta[n+1]$ with $h[n] * h[-n]$ gives coefficient of $\delta[n]$ equal to coeff of $\delta[n-1]$ in $h[n] * h[-n]$, which is -18. The answer is therefore

$$[8, 4, 4] \cdot [29, -18, -18] = 88.$$

13. Let Θ_1 and Θ_2 be independent RV's, each uniformly distributed in $[0, 2\pi]$. Define continuous-time processes $X(t)$ and $Y(t)$ by:

$$X(t) = 8\cos(t + \Theta_1) + 5, \quad -\infty < t < \infty$$

$$Y(t) = 8\cos(t + \Theta_2) + 5, \quad -\infty < t < \infty$$

(a) Compute $E[X(t)Y(t)]$.

Solution. $X(t)$ and $Y(t)$ are WSS, each having mean equal to 5. Therefore,

$$E[X(t)Y(t)] = E[X(t)]E[Y(t)] = \mu_X\mu_Y = 25.$$

(b) Compute $E[(X(t) + Y(t))^2]$.

Solution. The autocorrelation functions of $X(t)$ and $Y(t)$ are

$$\begin{aligned} R_X(\tau) &= 32\cos(\tau) + 25 \\ R_Y(\tau) &= 32\cos(\tau) + 25 \end{aligned}$$

$$E[(X(t) + Y(t))^2] = R_X(0) + R_Y(0) + 2\mu_X\mu_Y = 164.$$

(c) Compute $E[(X(t)Y(t))^2]$.

Solution.

$$E[(X(t)Y(t))^2] = E[X(t)^2]E[Y(t)^2] = R_X(0)R_Y(0) = 3249.$$

14. Continuous-time WSS process $X(t)$ has power spectral density as follows:

$$S_X(f) = \begin{cases} 50, & -5 \le f \le -3 \\ 50, & 3 \le f \le 5 \\ 0, & \text{elsewhere} \end{cases}$$

(a) Compute $E\left[\left(\int_{-\infty}^{t} X(u)du\right)^2\right]$.

Solution. You are passing $X(t)$ through a LTI system (an integrator) with frequency response function

$$H(f) = \frac{1}{j2\pi f}.$$

Letting $Y(t)$ be the output, you are computing P_Y.

$$P_Y = \int S_X(f)|H(f)|^2 df = 2\int_3^5 \frac{50}{(2\pi f)^2} df = \frac{10}{3\pi^2}.$$

(b) Compute $E\left[\left(\frac{dX(t)}{dt}\right)^2\right]$.

Solution. You are passing $X(t)$ through a LTI system (a differentiator) with frequency response function

$$H(f) = j2\pi f.$$

Letting $Y(t)$ be the output, you are computing P_Y.

$$P_Y = \int S_X(f)|H(f)|^2 df = 2\int_3^5 50(2\pi f)^2 df = \frac{39200\pi^2}{3}.$$

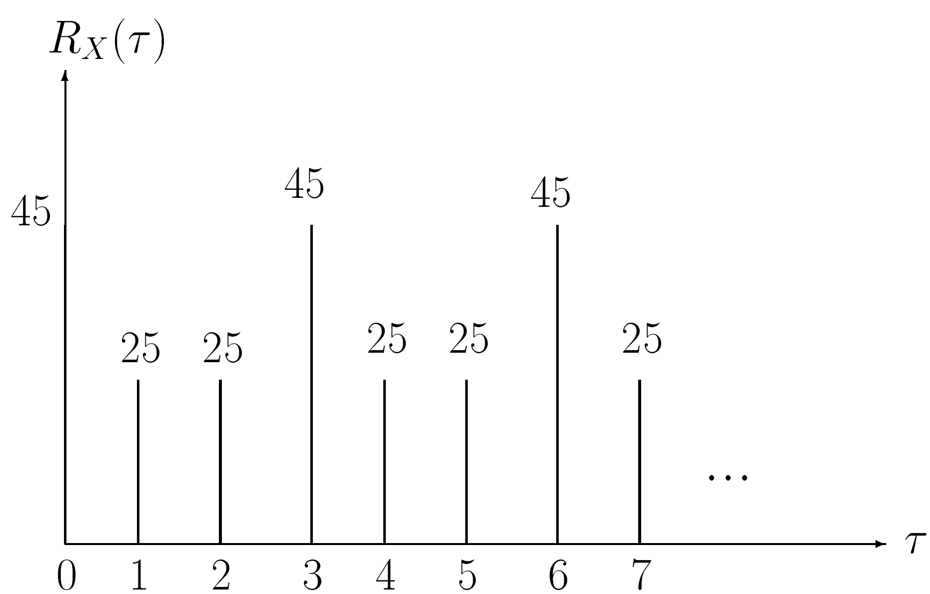
(c) Compute $E\left[(X(t) - X(t-1))^2\right]$.

Solution. The freq response is now

$$H(f) = 1 - \exp(-j2\pi f).$$

$$P_Y = \int S_X(f)|H(f)|^2 df = 2\int_3^5 50[2 - 2\cos(2\pi f)]df = 400.$$

15. A discrete-time WSS process X_n has autocorrelation function $R_X(\tau)$ which is periodic with period 3, and is plotted below for $\tau \geq 0$:



(a) X_2 is to be predicted from X_0 via the first order linear predictor of the form $\hat{X}_2 = AX_0$. Determine the value of the predictor coefficient A.

Solution.

$$A = \frac{R_X(2)}{R_X(0)} = 25/45.$$

(b) Determine the values of $\tau > 0$ for which X_τ can be perfectly predicted from X_0 using a first order linear predictor.

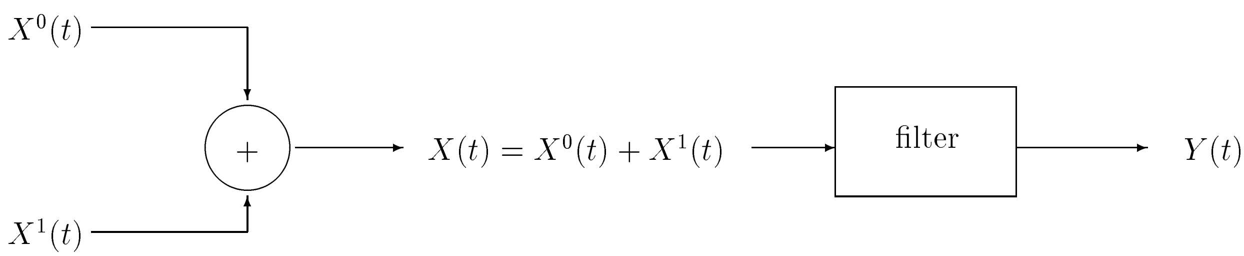
Solution. Identify the τ values where $|R_X(\tau)| = R_X(0)$. These are

$$\tau = 3, 6, 9, 12, \ldots.$$

(c) Given that $\mu_X^2 = \langle R_X(\tau) \rangle$, compute μ_X^2 and σ_X^2.

Solution.

$$\begin{aligned} \mu_X^2 &= (1/3)(45 + 25 + 25) = 95/3 \\ \sigma_X^2 &= R_X(0) - \mu_X^2 = 45 - 95/3 = 40/3 \end{aligned}$$



16. In the diagram above, $X^0(t), X^1(t)$ are 0-mean uncorrelated continuous-time random processes with

$$R_{X^0}(\tau) = 2.4e^{-10|\tau|}, \quad R_{X^1}(\tau) = \cos(20\tau).$$

The filter frequency response function is

$$H(f) = \frac{100}{j2\pi f + 20}.$$

(a) Compute $S_X(f)$.

Solution.

$$S_X(f) = S_{X^0}(f) + S_{X^1}(f) = \frac{48}{(2\pi f)^2 + 100} + (0.5)\delta(f - 10/\pi) + (0.5)\delta(f + 10/\pi).$$

(b) Compute P_X.

Solution.

$$P_X = R_X(\tau) = R_{X^0}(0) + R_{X^1}(0) = 3.4.$$

(c) Compute $S_Y(f)$.

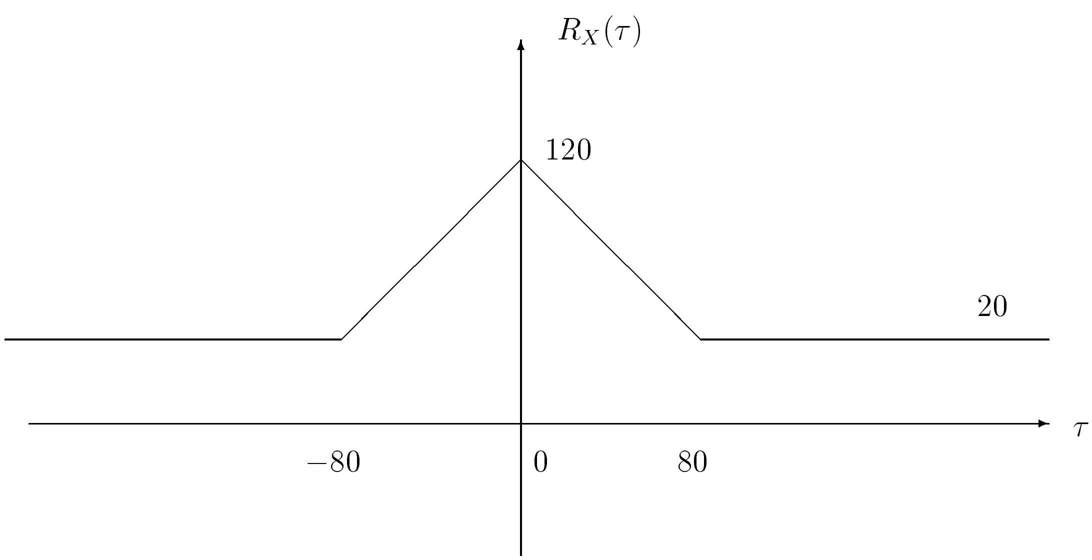
Solution.

$$\begin{aligned} S_Y(f) &= |H(f)|^2 S_X(f) \\ &= \frac{10000}{(2\pi f)^2 + 400} S_X(f) \\ &= \frac{480000}{(2\pi f)^2 + 100)((2\pi f)^2 + 400)} + 6.25\delta(f - 10/\pi) + 6.25\delta(f + 10/\pi) \end{aligned}$$

(d) Compute P_Y.

Solution.

$$\begin{aligned} P_Y &= \int_{-\infty}^{\infty} S_Y(f) df \\ &= 12.5 + \int_{-\infty}^{\infty} \left[\frac{1600}{(2\pi f)^2 + 100} - \frac{1600}{(2\pi f)^2 + 400} \right] df \\ &= 12.5 + 80 - 40 = 52.5 \end{aligned}$$



17. $X(t)$ is a Gaussian WSS ergodic process whose autocorrelation function $R_X(\tau)$ is plotted above.

(a) $E[X(500)]E[X(540)] =?$

Solution. The answer is $\mu_X^2 = 20$.

(b) $E[X(500)X(540)] =?$

Solution. The answer is $R_X(40) = 70$.

(c) Standard deviation of $X(500) =?$

Solution.

$$\sigma_X^2 = R_X(0) - \mu_X^2 = 120 - 20 = 100.$$

Therefore, the standard deviation is 10.

(d) $Cov(X(500), X(540)) =?$

Solution.

$$Cov(X(500), X(540)) = R_X(40) - \mu_X^2 = 70 - 20 = 50.$$

(e) Find smallest $t \geq 0$ for which $Cov(X(500), X(t)) = 0$.

Solution. The t we are looking for is the time t for which $R_X(t - 500) = 20$. Looking at the plot of $R_X(\tau)$, this clearly takes place at $t - 500 = 80$. Hence, $t = 580$.

(f) Find the standard deviation of $X(500) - X(540)$.

Solution.

$$Var(X(500) - X(540)) = 2\sigma_X^2 - 2Cov(X(500), X(540)) = 100.$$

Therefore, the standard deviation is 10.

(g) Assuming $\mu_X > 0$, find the constant D for which

$$P(X(500) > 15) = \int_D^\infty \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz.$$

Solution. $X(500)$ has mean $\sqrt{20}$ and standard deviation 10. Subtracting this mean and dividing by this standard deviation in order to convert to standard Gaussian, we see that

$$D = \frac{15 - \sqrt{20}}{10} = 1.053.$$

(h) Find the constant C for which

$$Cov(X(500), X(500) - CX(540)) = 0.$$

Solution.

$$\begin{aligned} Cov(X(500), X(500) - CX(540)) &= \sigma_X^2 - C * Cov(X(500), X(540)) \\ &= 100 - C * 50 \end{aligned}$$

It follows that $C = 2$.

18. $X(t)$ is a Gaussian WSS process satisfying $\mu_X = 1$ and $R_X(\tau) = \delta(\tau) + 1$. $Y(t)$ is the process defined by

$$Y(t) = \int_0^1 (s + t)X(s)ds, \quad -\infty < t < \infty$$

(a) Compute $E[Y(1)]$.

Solution. Since $E[X(s)] = 1$, we have

$$E[Y(1)] = \int_0^1 (s + 1)ds = 3/2$$

(b) Compute $E[Y(1)^2]$.

Solution. By the "double integral trick",

$$\begin{aligned} E[Y(1)^2] &= \int_0^1 \int_0^1 (s_1+1)(s_2+1)(1+\delta(s_1-s_2))ds_1 ds_2 \\ &= \left[\int_0^1 (s+1)ds\right]^2 + \int_0^1 (s+1)^2 ds = 55/12 \end{aligned}$$

(c) Determine the PDF $f(y)$ of $Y(1)$.

Solution. $Y(1)$ is Gaussian with mean 0 and variance is $55/12 - (3/2)^2 = 7/3$. This gives us

$$f(y) = \frac{1}{\sqrt{14\pi/3}} \exp(-3[y-3/2]^2/14)$$

19. A discrete-time WSS process X has power spectral density

$$S_X(f) = 2\delta(f+1/6)+2\delta(f-1/6)+(1.5)\delta(f+3/8)+(1.5)\delta(f-3/8), \quad -1/2 \le f \le 1/2.$$

Let Y be the discrete-time WSS process obtained by passing X through the discrete-time LTI filter with frequency response function

$$H(f) = \frac{5}{2+\exp(-j2\pi f)}$$

(a) Compute the power P_X generated by process X.

Solution. Integrating $S_X(f)$ from $-1/2$ to $1/2$ gives $P_X = 7$.

(b) Compute the power P_Y generated by process Y.

Solution.

$$|H(f)|^2 = \frac{25}{5+4\cos(2\pi f)}$$

$$|H(\pm 1/6)|^2 = 25/7.$$

$$|H(\pm 3/8)|^2 = 25/(5-2\sqrt{2}) = 11.51$$

$$P_Y = 2(25/7) + 2(25/7) + (1.5)(11.51) + (1.5)(11.51) = 48.8.$$

20. Let A, B be independent RV's each having mean 0 and variance 1. Let $X(t)$ be the continuous-time random process

$$X(t) = At + (2t-1)B, \quad -\infty < t < \infty$$

(a) Find the value of t for which $\mathrm{Var}[X(t)]$ is a minimum.

Solution.

$$\mathrm{Var}[X(t)] = t^2 + (2t-1)^2$$

Setting the first derivative equal to zero, one easily determines that $t = 2/5$.

(b) Find the value of t for which $E[X(t)X(t-1)]$ is a minimum.

Solution.

$$X(t)X(t-1) = \{At + (2t-1)B\}\{A(t-1) + (2t-3)B\}$$

$$E[X(t)X(t-1)] = t(t-1) + (2t-1)(2t-3)$$

Setting the derivative equal to zero yields $t = 9/10$.