Chapter 1 Solved Problems

1 Probability Model From Given Conditions

Problem 1.1: Three horses A, B, C run a race, and the winner is recorded. (There are no ties.) Horse A is twice as likely to win as Horse B, and Horse B is twice as likely to win as Horse C. Compute the probability each horse will win.

Solution. The sample space is

$$S = \{A, B, C\}$$

Let us compute $P(A), P(B), P(C)$ from the given information. Notice that

$$\begin{aligned} P(A) &= 2P(B) \\ P(B) &= 2P(C) \\ P(A) + P(B) + P(C) &= 1 \end{aligned}$$

There is only one solution to these equations, and it is

$$\begin{aligned} P(A) &= 4/7 \\ P(B) &= 2/7 \\ P(C) &= 1/7 \end{aligned}$$

2 Equiprobable Spaces

Problem 2.1: Three fair coins are tossed and it is recorded whether each of them is heads (H) or tails (T).

(a) Devise a sample space and probability model.

(b) What is the probability that there are exactly two heads?

(c) What is the probability that there are at most two heads?

Solution to (a).

$$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}.$$

The probability of each of these eight outcomes must be 1/8.

Solution to (b). The event we are interested in is

$$E_1 = \{HHT, HTH, THH\}.$$

Its probability is the number of outcomes in the event divided by 8, which gives

$$P(E_1) = 3/8.$$

Solution to (c). The event we are interested in is

$$E_2 = \{HHT, HTH, THH, TTH, THT, HTT, TTT\},$$

which has probability 7/8. Alternatively, the complement of E_2 is the outcome $\{HHH\}$, which has probability 1/8. Therefore,

$$P(E_2) = 1 - P(HHH) = 1 - (1/8) = 7/8.$$

Problem 2.2: Toss a pair of fair dice and see what numbers come up.

(a) Devise the sample space and probability model for this experiment.

(b) Compute the probability that the total of the numbers on the two dies is 7.

(c) Compute the probability that neither of the two numbers on the dies is bigger than 3.

Solution to (a). The sample space for this experiment is

$$\begin{aligned} S = \{ & (1,1),(1,2),(1,3),(1,4),(1,5),(1,6), \\ & (2,1),(2,2),(2,3),(2,4),(2,5),(2,6), \\ & (3,1),(3,2),(3,3),(3,4),(3,5),(3,6), \\ & (4,1),(4,2),(4,3),(4,4),(4,5),(4,6), \\ & (5,1),(5,2),(5,3),(5,4),(5,5),(5,6), \\ & (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \ \} \end{aligned}$$

Notice that S contains 36 outcomes. Since S is an equiprobable space, the probability of each outcome is 1/36.

Solution to (b). The event E_1 that the total is 7 may be written in subset form as

$$E_1 = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}.$$

Since we have an equiprobable space S, the probability of event E_1 is computed by dividing the number of outcomes in E_1 by 36. Therefore,

$$P(E_1) = 6/36 = 1/6.$$

Solution to (c). Let E_2 be the event that each die comes up no bigger than 3. Then

$$E_2 = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\},$$

and

$$P(E_2) = 9/36 = 1/4.$$

Problem 2.3: Mary, Fran, Rose, and Viola check their coats at a restaurant, and the coat checker returns their coats in random order. Compute the probability that each woman gets their own coat, and the probability that Mary gets her own coat.

Solution. We need to devise a sample space. Denote Mary's coat by C_M, Fran's coat by C_F, Rose's coat by C_R, and Viola's coat by C_V. We consider the outcomes of the experiment to be all possible permutations of these coats, with the first slot of the permutation filled by the coat that Mary gets, the second slot filled by the coat that Fran gets, the third slot filled by the coat that Rose gets, and the fourth slot filled by the coat that Viola gets. For example, the permutation (C_V, C_M, C_F, C_R) means that Mary get Viola's coat, Fran gets Mary's coat, Rose gets Fran's coat, and Viola gets Rose's coat. The sample space is of size $4! = 24$ and is equiprobable, with each outcome therefore having probability $1/24$. The event that each woman gets her own coat consists of the single outcome (C_M, C_F, C_R, C_V), and therefore the probability that each woman gets her own coat is $1/24$. The event that Mary gets her own coat consists of the $3! = 6$ outcomes in S which start with C_M. Therefore, the probability that Mary gets her own coat is $6/24$.

Problem 2.4: You are dealt a 5-card poker hand at random from a standard 52 card deck of playing cards. Compute the probability of getting a "three of a kind" poker hand.

Solution. The sample sample S is equiprobable, consisting of all 5-tuples of the form

$$(C_1, C_2, C_3, C_4, C_5),$$

where each C_i is a different card from the deck. The number of outcomes in this sample space is clearly

$$52 * 51 * 50 * 49 * 48.$$

The desired probability is therefore equal to the number of "3 of a kind" poker hands divided by $52 * 51 * 50 * 49 * 48$. To illustrate, a "3 of a kind" poker hand might take the form

$$(8, J, 8, 8, 6),$$

where the "3 of a kind"' cards are the 8's, and the other two cards appearing (the J card and the 6 card) are each "one of a kind". Here is a multiple-step procedure via which you can build all "3 of a kind" poker hands exactly once:

Step 1: Choose 3 out of the 5 positions in the poker hand in which to put the "3 of a kind"' cards. There are

$$\binom{5}{3} = 5!/(3!2!) = 10$$

ways to do this step.

Step 2: In the leftmost of the 3 positions chosen in Step 1, put a card. There are 52 ways to do this step.

Step 3: In the middlemost of the 3 positions chosen in Step 1, put a card of the same kind as the one selected in Step 2. There are 3 ways to do this step. (For example, if the Step 2 card is 8 of hearts, then the middlemost card has to be either 8 of diamonds, clubs, or spades.)

Step 4: In the rightmost of the 3 positions chosen in Step 1, put a card of the same kind as the one selected in Step 2. There are 2 ways to do this step.

Step 5: In the leftmost of the two positions not chosen in Step 1, put a card of a different kind than the card selected in Step 2. There are 48 ways to do this step.

Step 6: In the rightmost of the two positions not chosen in Step 1, put a card of a different kind than the Step 2 card and the Step 5 card. There are 44 ways to do this.

The number of 3 of a kind poker hands is therefore

$$10 * 52 * 3 * 2 * 48 * 44.$$

The probability of a 3 of a kind poker hand is therefore

$$\frac{10 * 52 * 3 * 2 * 48 * 44}{52 * 51 * 50 * 49 * 48}.$$

Problem 2.5: Compute the probability of getting a "two pair" poker hand.

Solution. Each two pair hand can be constructed from the following steps exactly once:

Step 1: Choose 2 out of the 5 positions in the poker hand in which to put the first pair. In the leftmost of these two positions, put any card from the deck. In the other position, put a card of the same kind. There are

$$\binom{5}{2} * 52 * 3 = 10 * 52 * 3$$

ways to perform this step.

Step 2: Choose 2 of the 3 remaining positions in the poker hand in which to put the second pair. In the leftmost of these two positions, put any card from the deck which is of a kind different from the kind of the first pair. In the other position, put a card of the same kind. There are

$$\binom{3}{2} * 48 * 3 = 3 * 48 * 3$$

ways to perform this step.

Step 3: In the fifth and final position in the poker hand, put any card of a kind different than the previous two kinds. There are 44 possibilities for this card.

The total number of two pair poker hands is therefore

$$10 * 52 * 3 * 3 * 48 * 3 * 44.$$

Divide this by

$$52 * 51 * 50 * 49 * 48, \tag{1}$$

the total number of poker hands, to obtain the probability of a two pair poker hand.

Problem 2.6: Compute the probability of getting a "full house" poker hand.

Solution. Each full house hand (3 of one kind, 2 of another kind) can be constructed from the following steps exactly once:

Step 1: Choose 3 of the 5 positions in the poker hand in which to put the 3 of a kind. Fill the leftmost of these 3 positions with any card, and fill the other two positions with cards of the same kind. There are

$$10 * 52 * 3 * 2$$

ways to do this step.

Step 2: In the leftmost of the two remaining positions in the poker hand, put any card of a kind different than the kind used in the 3 of a kind, and in the other position put a card of the same kind. There are

$$48 * 3$$

ways to do this step.

The total number of full house poker hands is therefore

$$10 * 52 * 3 * 2 * 48 * 3.$$

Dividing by (1), you obtain the probability of getting a full house.

Problem 2.7: N randomly selected people are placed in a room.

(a) Compute the probability that at least two of them have the same birthday. (For simplicity, do not consider the situation in which a person has a birthday in a leap year.)

(b) Show that if $N = 23$, there is a better than 50-50 chance that two or more of the people will have the same birthday.

Solution to (a). The sample space S consists of all N-tuples

$$(B_1, B_2, \cdots, B_N)$$

in which B_i is the birthday of person i. The size of S is 365^N. Let E be the event that no two people have the same birthday. Then:

- The probability we want is $1 - P(E)$.
- We can model E as the set of all

$$(B_1, B_2, \cdots, B_N)$$

in which the B_i are all different. There are clearly

$$365 * 364 * 363 * \cdots * (365 - N + 1)$$

such N-tuples (fill first position 365 ways, second position 364 ways, etc.).

We have an equiprobable sample space. Therefore,

$$P(E) = \left(\frac{365 * 364 * 363 * \cdots * (365 - N + 1)}{365^N} \right).$$

We conclude that the probability that at least two of the N people have the same birthday is

$$1 - P(E) = 1 - \left(\frac{365 * 364 * 363 * \cdots * (365 - N + 1)}{365^N} \right).$$

Solution to (b). For $N = 23$, we compute the probability found at the end of the solution to (a) using the following Matlab script:

```
N=23;
num=(365-N+1):365;
den=365*ones(1,N);
ratio=num./den;
probability=1-prod(ratio)
```

Running this script, the reader will conclude that this probability is 0.5073. Thus, if you have 23 randomly selected people, there is indeed a better than 50-50 chance that two (or more) of them will have the same birthday.

3 Probability Models Using a Tree

Problem 3.1: You have three coins. Coin 1 is a fair coin. Coin 2 has $P(H) = 0.4$. Coin 3 has $P(H) = 0.55$. Consider the following 3-step experiment. On Step 1, coin 1 is tossed and the outcome (H or T) is recorded. On Step 2, coin 2 is tossed and the outcome (H or T) is recorded. On Step 3, coin 3 is tossed and the outcome (H or T) is recorded.

(a) Find the probability model for this experiment.

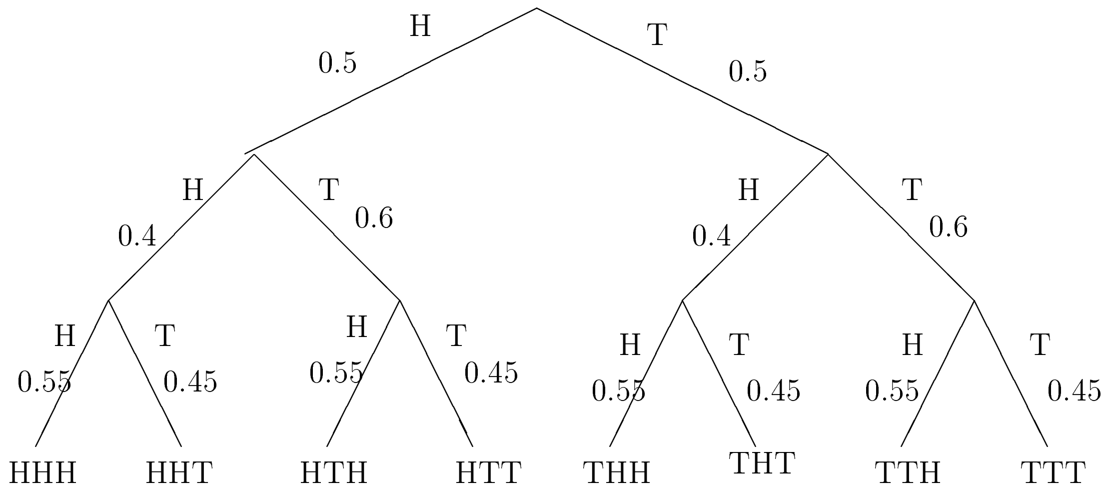
(b) Find the probability of exactly two heads.

(c) Find the probability of at least one head.

Solution. The sample space is:

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

The following tree models these outcomes as labels on the leaves of a tree:



Multiplying along the 8 root-to-leaf paths, you get the probability model for this experiment:

$$P(HHH) = (0.5)(0.4)(0.55) = 0.11$$
$$P(HHT) = (0.5)(0.4)(0.45) = 0.09$$
$$P(HTH) = (0.5)(0.6)(0.55) = 0.165$$
$$P(HTT) = (0.5)(0.6)(0.45) = 0.135$$
$$P(THH) = (0.5)(0.4)(0.55) = 0.11$$
$$P(THT) = (0.5)(0.4)(0.45) = 0.09$$
$$P(TTH) = (0.5)(0.6)(0.55) = 0.165$$
$$P(TTT) = (0.5)(0.6)(0.45) = 0.135$$

Let E be the event that there are exactly two heads. Then,

$$E = \{HHT, HTH, THH\}.$$

We have

$$P(E) = P(HHT) + P(HTH) + P(THH) = 0.09 + 0.165 + 0.11 = 0.365$$

There are two ways to compute the probability that there is at least one head. One way is to add up the probabilities of the 7 outcomes comprising this event:

$$P(HHH) + P(HTH) + P(THH) + P(HHT) + P(TTH) + P(THT) + P(HTT)$$

The other way is simply to subtract $P(TTT)$ from 1. The second way is easier:

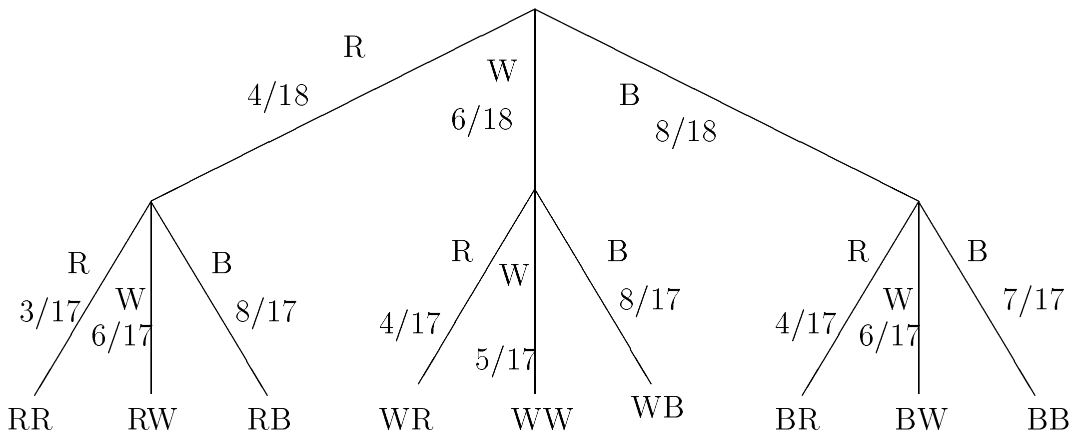
$$P(\text{at least one head}) = 1 - P(TTT) = 1 - 0.135 = 0.865.$$

Problem 3.2: An urn contains 4 red, 6 white, and 8 blue balls. Consider the following two-step experiment. In Step 1, a ball is selected at random from the urn and its color is noted. Then, the ball is set aside. In Step 2, a second ball is selected at random from the urn and its color is noted. (This is called sampling without replacement.) The outcome of this experiment consists of the sequence of two colors obtained in the order obtained. Find the probability model for this experiment.

Solution. The sample space for this experiment is

$$S = \{RR, RW, RB, WR, WW, WB, BR, BW, BB\}.$$

The probability model can be obtained from the following tree:

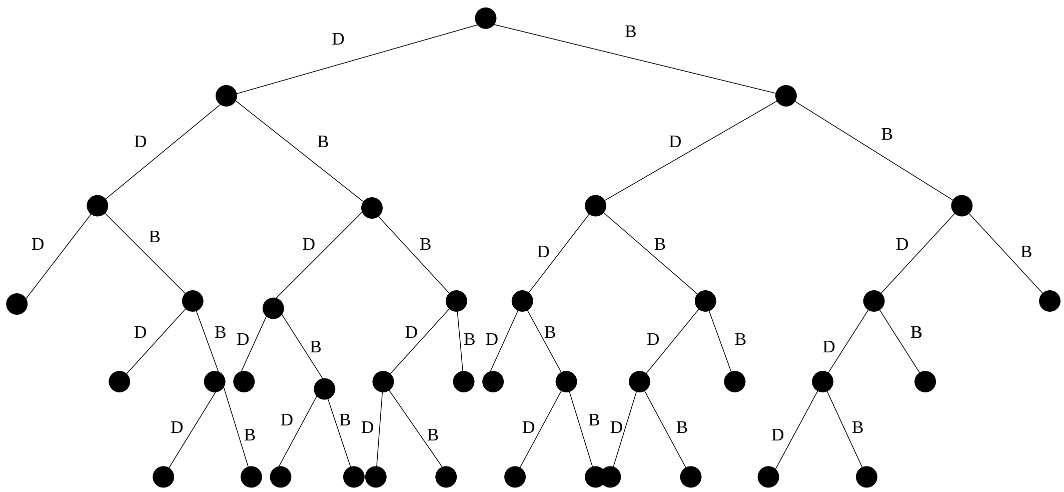


For example, the RR, RW, and RB paths in the tree correspond to choosing the first ball to be red, and the second ball is then red, white, or blue, respectively. You have 4 chances in 18 of drawing a red ball on the first draw; therefore, the probability 4/18 labels the top level R branch in the tree. Once a red ball is removed from the urn after the first draw, the urn then contains 17 balls broken down as 3 red, 6 white, and 8 blue balls, so the probability labels for 3 second level branches in the tree (corresponding to the second ball being R,W,B) must be $3/17, 6/17, 8/17$, respectively. In this way, probabilities are assigned to all 12 branches of the tree. Multiplying the probabilities along each of the 9 paths in the tree yields the following probability model for the 9 outcomes in the sample space:

$$P(RR) = (4/18)(3/17)$$
$$P(RW) = (4/18)(6/17)$$
$$P(RB) = (4/18)(8/17)$$
$$P(WR) = (6/18)(4/17)$$
$$P(WW) = (6/18)(5/17)$$
$$P(WB) = (6/18)(8/17)$$
$$P(BR) = (8/18)(4/17)$$
$$P(BW) = (8/18)(6/17)$$
$$P(BB) = (8/18)(7/17)$$

Problem 3.3: The Dodgers and Braves play a best 3 out of 5 playoff series. (The first team to win 3 games wins the series.) We assume that the two teams are equally matched. Compute the probability that the series will take exactly four games.

Solution. The following tree represents all the possibilities here:



From the tree (following all paths from the root to the terminal vertices) there are 20 outcomes in the sample space S:

$$S = \{DDD, DDBD, DDBBD, DDBBB, DBDD, DBDBD,$$

$$DBDBB, DBBDD, DBBDB, DBBB, BDDD, BDDBD, BDDBB,$$

$$BDBDD, BDBDB, BDBB, BBDDD, BBDDB, BBDB, BBB\}$$

Each branch of the tree should be assigned a probability label of 1/2. Multiplying along each root-to-leaf path, we then get the following probability model for this experiment:

$P[DDD] = 1/8$	$P[DDBD] = 1/16$	$P[DDBBD] = 1/32$	$P[DDBBB] = 1/32$
$P[DBDD] = 1/16$	$P[DBDBD] = 1/32$	$P[DBDBB] = 1/32$	$P[DBBDD] = 1/32$
$P[DBBDB] = 1/32$	$P[DBBB] = 1/16$	$P[BDDD] = 1/16$	$P[BDDBD] = 1/32$
$P[BDDBB] = 1/16$	$P[BDBDD] = 1/16$	$P[BDBDB] = 1/32$	$P[BDBB] = 1/16$
$P[BBDDD] = 1/32$	$P[BBDDB] = 1/32$	$P[BBDB] = 1/16$	$P[BBB] = 1/8$

The event E that the series takes four games is:

$$E = \{DDBD, DBDD, DBBB, BDDD, BDBB, BBDB\}$$

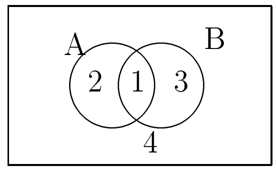
Therefore,

$$P(E) = 6/16 = 3/8.$$

4 Venn Diagram Reasoning

Problem 4.1: Let A, B be events associated with the same random experiment. If the probability that at least one of the two events occurred is 0.7, and the probability that at least one of the two events did not occur is 0.6, compute the probability that exactly one of the two events occurred.

Solution.



It is given that

$$P[A \cup B] = P(1) + P(2) + P(3) = 0.7$$
$$P[A^c \cup B^c] = P(2) + P(3) + P(4) = 0.6$$

Then

$$\begin{aligned} P[\text{exactly one occurs}] &= P(2) + P(3) = \\ (P(1) + P(2) + P(3)) - P(1) &= 0.7 - (1 - P(2) - P(3) - P(4)) = 0.3 \end{aligned}$$

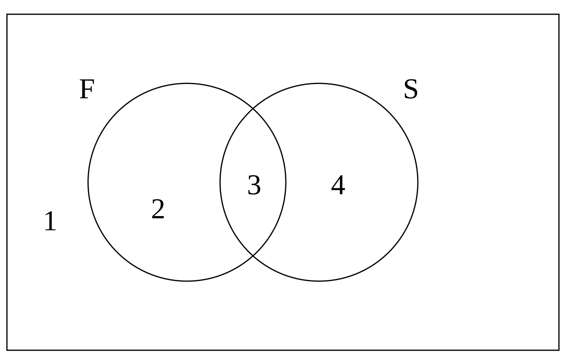
Problem 4.2: At a certain college, 30% of the students take French, 20% of the students take Spanish, and 10% of the students take both of these languages. A student is selected at random. Compute the probability that the student

(a) takes at least one of the two languages

(b) takes neither of the two languages

(c) takes exactly one of the two languages.

Solution. Refer to the Venn Diagram which follows.



F is the event that the student takes French and S is the event that the student takes Spanish. The entire sample space is partitioned into regions 1,2,3,4, where:

$$
\begin{aligned}
1 &= F^c \cap S^c = \{\text{takes neither}\} \\
2 &= F \cap S^c = \{\text{takes } F \text{ but not } S\} \\
3 &= F \cap S = \{\text{takes both}\} \\
4 &= F^c \cap S = \{\text{takes } S \text{ but not } F\}
\end{aligned}
$$

One condition that we automatically have concerning the events 1,2,3,4 is that their probabilities add up to one. If we are given three other conditions concerning these events, then we will have a total of four equations in the four unknowns $P(1), P(2), P(3), P(4)$ which can be solved simultaneously to uniquely determine these four probabilities. Once these four probabilities are known, one can answer any probability question about the above Venn diagram.

From the Venn diagram, it is clear that

$$
\begin{aligned}
P(3) &= P(F \cap S) = 0.1 \\
P(2) &= P(F) - P(3) = 0.3 - 0.1 = 0.2 \\
P(4) &= P(S) - P(3) = 0.2 - 0.1 = 0.1 \\
P(1) &= 1 - (P(2) + P(3) + P(4)) = 0.6
\end{aligned}
$$

We conclude that:

$$
\begin{aligned}
P(\text{takes at least one}) &= P(F \cup S) = P(2) + P(3) + P(4) = 0.4 \\
P(\text{takes neither}) &= P(1) = 0.6 \\
P(\text{takes exactly one}) &= P(2) + P(4) = 0.3
\end{aligned}
$$

Problem 4.3: Suppose three events A, B, C satisfy:

$$P(A) = P(B) = P(C) = 0.5$$

$$P(A \cap B) = P(A \cap C) = P(B \cap C) = 0.3$$
$$P(A \cap B \cap C) = 0.18$$

Compute the following probabilities:

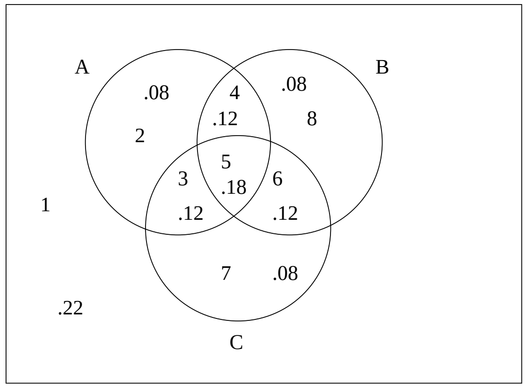
(a) P[at least one of A,B,C occur]

(b) P[none of A,B,C occur]

(c) P[exactly one of A,B,C occur]

(d) P[exactly two of A,B,C occur]

Solution. Using the Venn diagram



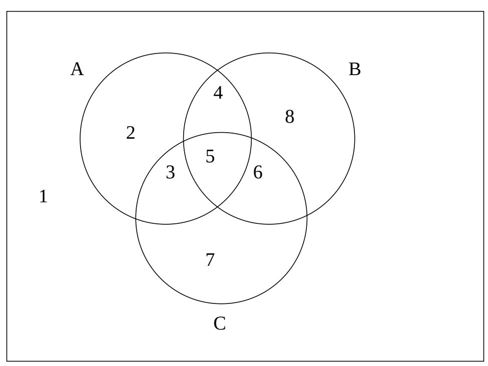
we obtain

```
P[at least one of A,B,C occur]=P(2)+P(3)+P(4)+P(5)+P(6)+P(7)+P(8)=0.78
P[none of A,B,C occur] = P(1)= 0.22
P[exactly one of A,B,C occur] = P(2)+P(7)+P(8)= 0.24
P[exactly two of A,B,C occur] = P(3)+P(4)+P(6) = 0.36
```

Problem 4.4: Three independent events A, B, C have probabilities

$$P(A) = 0.2, \quad P(B) = 0.3, \quad P(C) = 0.4.$$

Determine the probabilities of each of the regions 1-8 in the following Venn Diagram:



Solution.

$$\begin{aligned}
P(1) &= P(A^c \cap B^c \cap C^c) = P(A^c)P(B^c)P(C^c) = (0.8)(0.7)(0.6) \\
P(2) &= P(A \cap B^c \cap C^c) = P(A)P(B^c)P(C^c) = (0.2)(0.7)(0.6) \\
P(3) &= P(A \cap B^c \cap C) = P(A)P(B^c)P(C) = (0.2)(0.7)(0.4)
\end{aligned}$$

$$
\begin{aligned}
P(4) &= P(A \cap B \cap C^c) = P(A)P(B)P(C^c) = (0.2)(0.3)(0.6) \\
P(5) &= P(A \cap B \cap C) = P(A)P(B)P(C) = (0.2)(0.3)(0.4) \\
P(6) &= P(A^c \cap B \cap C) = P(A^c)P(B)P(C) = (0.8)(0.3)(0.4) \\
P(7) &= P(A^c \cap B^c \cap C) = P(A^c)P(B^c)P(C) = (0.8)(0.7)(0.4) \\
P(8) &= P(A^c \cap B \cap C^c) = P(A^c)P(B)P(C^c) = (0.8)(0.3)(0.6)
\end{aligned}
$$

5 Conditional Probabilities

Problem 5.1: Two cards are drawn from a standard 52 card deck without replacement. Compute the following probabilities:

(a) Prob second card is a king given first card is a king.

(b) Prob second card is a king given the first card is not a king.

(c) Prob second card is a king.

(d) Prob second card is a heart given first card is the king of hearts.

(e) Prob second card is a heart given first card is a king but not a heart.

(f) Prob second card is a heart.

(g) Prob second card is a heart given first card is a king.

Solution to (a). There are 51 cards to chose from, of which 3 are kings. So the answer is 3/51.

Solution to (b). There are 51 cards to chose from, of which 4 are kings. So the answer is 4/51.

Solution to (c). Let $K1$ be the event first card is a king. Let K^c1 be the event the first card is not a king. Let $K2$ be the event the second card is a king. Then

$$P(K2) = P(K2|K1)P(K1) + P(K2|K^c1)P(K^c1) = (3/51)(1/13) + (4/51)(12/13) = 1/13.$$

Solution to (d). 12/51.

Solution to (e). 13/51.

Solution to (f). The approach is similar to that of part (c). The answer will turn out to be the same as prob of getting a heart on first draw, which is 1/4.

Solution to (g). H2 is the event you get heart on second draw and H1 that you get heart on first draw. Then

$$P(H2 \cap K1 \cap H1) = P(K1 \cap H1)P(H2|K1 \cap H1) = (1/52) * (12/51)$$

$$P(H2 \cap K1 \cap H^c1) = P(K1 \cap H^c1)P(H2|K1 \cap H^c1) = (3/52) * (13/51)$$

The probability $P(H2 \cap K1)$ is the sum of these, which is 1/52. We conclude that

$$P(H2|K1) = P(H2 \cap K1)/P(K1) = (1/52)/(1/13) = 1/4.$$

(As a by-product, since $P(H2|K1)$ and $P(H2)$ are both 1/4, the events $H2$ and $K1$ are independent events, even though they are connected with dependent draws!)

Problem 5.2: There are three children in a family. Assume that each child is equally likely to be a boy or girl, independently of the other two children. Using the reduced sample space approach, compute the probability that there is at least one boy given that there is at least one girl.

Solution. The sample space is

$$\{BBB, BBG, BGB, GBB, GGB, GBG, BGG, GGG\},$$

each of the 8 outcomes having prob 1/8. The reduced sample space is

$$\{BBG, BGB, GBB, GGB, GBG, BGG, GGG\}.$$

In the reduced sample space, the outcomes are still equally likely, but since there are only 7 outcomes, they must each have probability 1/7 instead of 1/8. Six of the outcomes in the reduced sample space correspond to "at least one boy." The answer is therefore 6/7.

Problem 5.3: A month of the year is picked randomly. Then, a day from that month is picked randomly. (Assume the year is not a leap year.)

(a) Given that the month selected is a 31-day month, what is the conditional probability that the day selected is between the 10th and the 20th (including the 10th and the 20th)?

(b) Given that the month selected is a 30-day month, what is the conditional probability that the day selected is between the 10th and the 20th?

(c) Given that the month selected is February, what is the conditional probability that the day selected is between the 10th and the 20th?

(d) What is the probability that the day selected is between the 10th and the 20th?

Solution to (a). 11/31.
Solution to (b). 11/30.
Solution to (c). 11/28.
Solution to (d). Use the theorem on total probability. Take weighted average of the cond probs in (a),(b),(c) with respect to the prob 31-day month is selected, prob 30-day month is selected, and the prob February is selected, respectively. Since there are seven 31-day months and four 31-day months, the answer is

$$(11/31) * (7/12) + (11/30) * (4/12) + (11/28) * (1/12).$$

Problem 5.4: An urn contains 6 red, 6 white, and 6 blue balls. Four balls are selected at random from the urn, one after the other, without replacement. It is observed that the second, third, and fourth balls selected are white, white, red, in that order. Given this information, was the first ball most likely to have been red, white, or blue?

Solution. Let Ri ($i = 1, 2, 3, 4$) denote the event of getting a red ball on draw i. Similarly, we let Wi and Bi denote the events of getting a white ball and a blue ball on draw i, respectively. We want to determine which of the following 3 numbers is the greatest:

$$P(R1|W2 \cap W3 \cap R4), \quad P(W1|W2 \cap W3 \cap R4), \quad P(B1|W2 \cap W3 \cap R4) \tag{2}$$

If we multiply these three numbers by $P(W2 \cap W3 \cap R4)$, we obtain the 3 numbers

$$P(R1 \cap W2 \cap W3 \cap R4), \quad P(W1 \cap W2 \cap W3 \cap R4), \quad P(B1 \cap W2 \cap W3 \cap R4). \tag{3}$$

Therefore, if we can determine which of the three numbers (3) is the greatest, the corresponding number in (2) will be the greatest. By the multiplication theorem, we have

$$\begin{aligned}
P(R1 \cap W2 \cap W3 \cap R4) &= P(R1)P(W2|R1)P(W3|R1 \cap W2)P(R4|R1 \cap W2 \cap W3) \\
&= (6/18)(6/17)(5/16)(5/15) \\
P(W1 \cap W2 \cap W3 \cap R4) &= P(W1)P(W2|W1)P(W3|W1 \cap W2)P(R4|W1 \cap W2 \cap W3) \\
&= (6/18)(5/17)(4/16)(6/15) \\
P(B1 \cap W2 \cap W3 \cap R4) &= P(B1)P(W2|B1)P(W3|B1 \cap W2)P(R4|B1 \cap W2 \cap W3) \\
&= (6/18)(6/17)(5/16)(6/15)
\end{aligned}$$

The third of these numbers is the biggest. Therefore, the first ball selected most likely was blue.

Problem 5.5: A game show host named Monte Hall hides a new car behind one of three doors; the door he selects is random. A contestant is led in and asked to guess which door contains the car. However, the door guessed by the contestant is not opened; instead, Monte Hall opens a door different from the door guessed by the contestant and different from the door containing the car (there is either exactly one choice for this door or two choices—if there are two choices for this door, it does not matter which one Monte picks). The contestant then opens the only remaining door (the door which was not initially guessed by the contestant and which is not the door selected by Monte Hall). What is the probability that the contestant opens the door containing the car?

Solution. Let E be the event that the door guessed by the customer contains the car. Let F be the event that the customer opens the door containing the car. Since the door opened by the contestant is never the same as the door he/she guessed initially, we must have

$$P(F|E) = 0.$$

On the other hand,

$$P(F|E^c) = 1,$$

because if E^c occurs, neither the door opened by Monte nor the door initially guessed by the contestant contain the car, and therefore the only remaining door must contain the car. Applying the theorem on total probability,

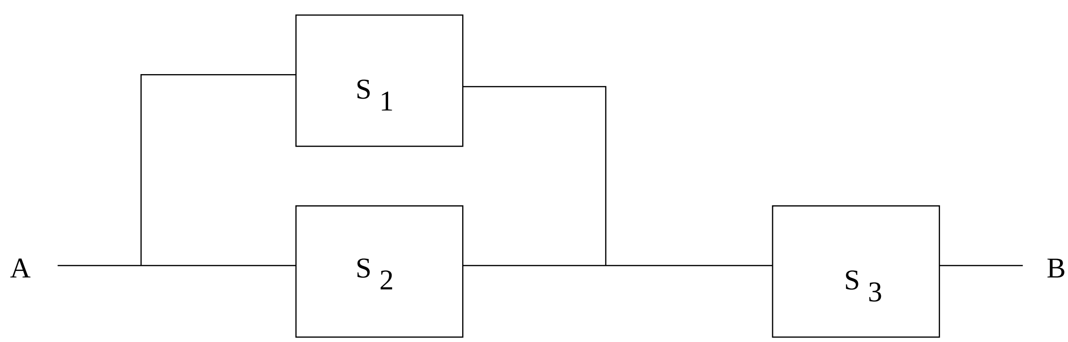
$$P(F) = P(F|E)P(E) + P(F|E^c)P(E^c) = 0 * (1/3) + 1 * (2/3) = 2/3.$$

Therefore, by taking into account the information that Monte gives the contestant, the contestant is able to double his/her chances of winning the car.

Problem 5.6: This problem is a classic problem called the "Prisoner's Dilemma". A prisoner is given two identical unmarked urns, 50 white balls, and 50 black balls. He/she is asked to distribute the 100 balls among the two urns. The urns are to be taken away, shuffled, and then returned, upon which the prisoner will be asked to choose an urn at random and then select a ball at random from that urn. If the ball is white, the prisoner will be set free. How should the prisoner distribute the 100 balls among the two urns to maximize his/her chances of being set free?

Solution. If the prisoner distributes the balls in the right way, then his/her probability of selecting the white ball will be 74/99. From this answer, see if you can figure out how the balls are distributed. (This problem is an exercise in the "theorem on total probability", such as was used in the solution to the Monte Hall Problem.)

6 Relay Circuit Problems



Problem 6.1: Consider the relay circuit above for transferring a signal from point A to point B. The components S_1, S_2, S_3 are switches which operate independently of each other. Each switch has only two possible states, "on" or "off". Let p be the probability that switch S_i is on (the same for all three switches).

(a) Compute the probability that the circuit will provide a connection from point A to point B (as a function of p).

(b) Determine p so that the circuit will provide the A to B connection with probability 0.90.

Solution to (a). For $i = 1, 2, 3$, let S_i denote the event that switch S_i is on. There are eight possible "states" of the system:

state	probability
$S_1 \cap S_2 \cap S_3$	p^3
$S_1^c \cap S_2 \cap S_3$	$p^2(1-p)$
$S_1 \cap S_2^c \cap S_3$	$p^2(1-p)$
$S_1 \cap S_2 \cap S_3^c$	$p^2(1-p)$
$S_1^c \cap S_2^c \cap S_3$	$p(1-p)^2$
$S_1^c \cap S_2 \cap S_3^c$	$p(1-p)^2$
$S_1 \cap S_2^c \cap S_3^c$	$p(1-p)^2$
$S_1^c \cap S_2^c \cap S_3^c$	$(1-p)^3$

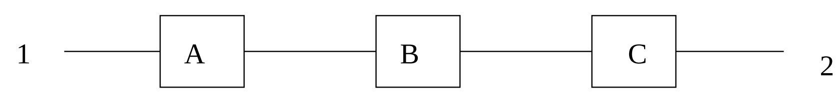
The first three states listed are precisely the system states for which current can flow from point A to point B. So, the probability that current can flow can point A to point B is

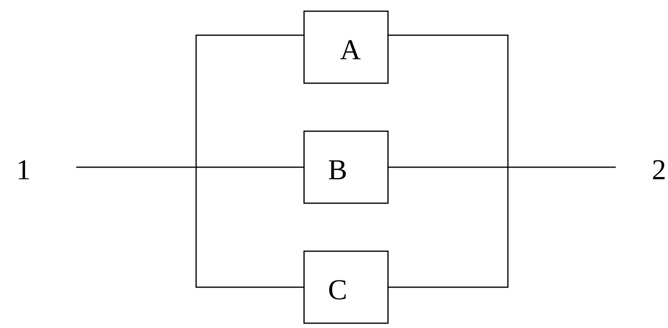
$$p^3 + 2p^2(1-p) = 2p^2 - p^3$$

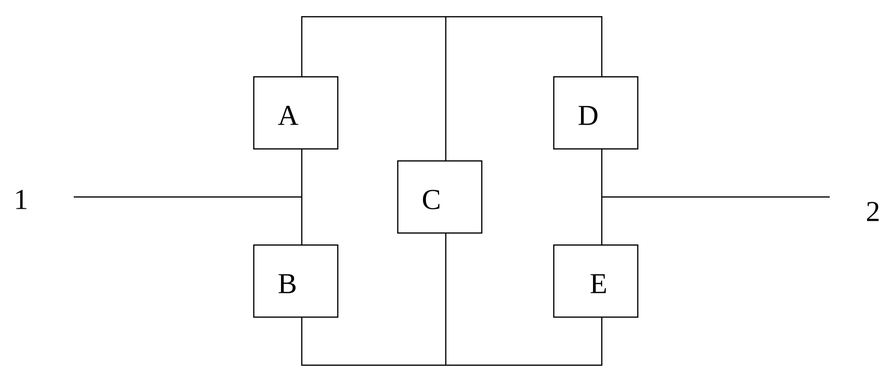
Solution to (b). Set

$$2p^2 - p^3 = 0.90,$$

and then use Matlab function "`roots`" to solve for p. You get $p = 0.9077$.







Problem 6.2: In each of the three relay circuits above, the switches $A, B, C, \ldots$ are 90% reliable and work independently. In each case, compute the probability that the relay circuit will relay a signal from point 1 to point 2.

Solution for top circuit. If R is a switch, let R^{+1} denote the event that the switch works, and let R^{-1} denote the event that the switch fails. Notation $A^{+1}B^{+1}C^{+1}$ denotes the event that the three switches A, B, C all work; this is the event that the circuit works for the three switches connected in series. The probability of this event is what we want. By "independent trials", this probability can be computed by multiplication as

$$P(A^{+1}B^{+1}C^{+1}) = P(A^{+1})P(B^{+1})P(C^{+1}) = (0.90)^3 = 0.729$$

Solution for middle circuit. For the three switches connected in parallel, the event $A^{+1} \cup B^{+1} \cup C^{+1}$ is the event that the circuit will work. The probability of this event is

$$1 - P(A^{-1}B^{-1}C^{-1}) = 1 - (0.1)^3 = 0.9990$$

Solution for bottom circuit. The "feedforward paths" are AD, BE, ACE, and BCD. We are interested in the union of these:

$$P(AD \cup BE \cup ACE \cup BCD).$$

To compute this probability we can take sums and differences of probabilities of intersections of these 4 events. Let S_1, S_2, S_3, S_4 be the sum of probabilities of intersections one at a time, two at a time, three at a time, and four at a time, respectively; S_1 contains 4 terms, S_2 contains 6 terms, S_3 contains 4 terms, and S_4 contains 1 term. We have

$$S_1 = P(AD) + P(BE) + P(ACE) + P(BCD) = 2(0.9)^2 + 2(0.9)^3 = 3.0780$$

$$\begin{aligned}
S_2 &= P(ADBE) + P(ADCE) + P(ADBC) + P(BEAC) + P(BECD) + P(ACEBD) \\
&= 5(0.9)^4 + (0.9)^5 = 3.8710 \\
S_3 &= P(ADBEC) + P(ADBEC) + P(ADCEB) + P(BEACD) = 4(0.9)^5 = 2.3620 \\
S_4 &= P(AD \cap BE \cap ACE \cap BCD) = P(ABCDE) = (0.9)^5 = 0.5905
\end{aligned}$$

We conclude that

$$\begin{aligned}
P(AD \cup BE \cup ACE \cup BCD) &= S_1 - S_2 + S_3 - S_4 \\
&= 3.0780 - 3.8710 + 2.3620 - 0.5905 = 0.9785
\end{aligned}$$

Recitation 2 gives a Matlab simulation of this circuit. The reader can run that simulation to see if the estimated probability that the circuit works is roughly 0.9785.

7 Independent Trials

Problem 7.1: John and Mary play a game. They throw a fair die alternately, with John starting first. The winner of the game is the first person to throw a "4". Compute the probability that John wins the game.

Solution 1. The sample space for this experiment may be written

$$S = \{4, 4^c4, 4^c4^c4, 4^c4^c4^c4, 4^c4^c4^c4^c4, \ldots\}$$

(an infinite sample space). In the outcomes in S, the notation 4 denotes that a die toss resulted in 4, whereas the notation 4^c denotes that a die toss resulted in one of the numbers $1, 2, 3, 5, 6$. The probability of each outcome is computed using independent trials. For example,

$$P[4^c4^c4^c4] = (5/6)^3(1/6)$$

The event that John wins is the event

$$\{4, 4^c4^c4, 4^c4^c4^c4^c4, \ldots\}$$

The probability that John wins is then

$$\sum_{i=0}^{\infty}(1/6)(5/6)^{2i} = 6/11$$

(We used here the fact that $\sum_{i=0}^{\infty} r^i = 1/(1-r)$ for $-1 < r < 1$.)

Solution 2. Here is a more clever (and easier) solution, using conditional probabilities. Let p be the probability John wins the game. Let E be the event that the die flip is 4 on the first flip, and let J be the event that John wins the game. Then

$$\begin{aligned} P(J|E) &= 1 \\ P(J|E^c) &= 1 - p \end{aligned}$$

($P(J|E^c) = 1 - p$ because if the first flip is not 4, then the succeeding flips are just like starting the game over again with the roles of John and Mary reversed.) We have, by the "theorem on total probability,"

$$P(J) = P(J|E)P(E) + P(J|E^c)P(E^c),$$

which reduces to the equation

$$p = (1/6) + (1-p)(5/6).$$

Solving this equation yields $p = 6/11$.

Problem 7.2: In a best 4 out of 7 championship series between two sports teams, the series ends as soon as one of the teams wins 4 games. Assume that the two teams are equally matched and that wins or losses in the different games occur independently. Compute the probability that the team that wins the series lost the first game.

Solution. The teams are denoted 1 and 2. We denote possible ways to play out the series as certain words formed from 1's and 2's. For example, the outcome 1222112 means the series went 7 games, with team 1 winning the first, fifth, and sixth games, and team 2 winning all the other games. There are two ways to construct outcome "words" in which the game 1 and last game winners are different:

- find all words with three 2's and no more than two 1's, and put a 1 in front of all these and a 2 behind; or
- find all words with three 1's and no more than two 2's, and put a 2 in front of all these and a 1 behind.

By symmetry, the probability assigned to these two sets of words is the same, so we can just compute the probability for one of the sets and multiply by 2. The words in the first set are (before tacking on the 1 in front and the 2 behind)

$$222$$

$$2221, 2212, 2122, 1222$$

$$22211, 22112, 21122, 11222, 12221, 12122, 22121, 12212, 21221, 21212$$

The probability we want is now clearly

$$2*(1/2)*(1/2)*[(1/8)+4(1/16)+10(1/32)] = 11/32.$$

Problem 7.3: Player 1 and Player 2 play the game called Rock, Scissors, Paper. On each trial each player indicates either Rock, Scissors, or Paper. Rock beats Scissors, Scissors beats Paper, and Paper beats Rock. If on a trial both players choose the same thing, they must go on to the next trial. The game is over and the winner declared, as soon as the first trial occurs in which they don't choose the same thing. Suppose Player 1 chooses Rock with probability 1/4, Scissors with probability 1/2, and Paper with probability 1/4. Suppose Player 2 chooses Rock with probability 1/4, Scissors with probability 1/4, and Paper with probability 1/2. What is the probability that Player 1 will win the game of Rock, Scissors, Paper?

Solution. Let 1 be the event that Player 1 wins, and let E be the event that the game just takes one trial. On each trial, there are nine possibilities

$$RS, SR, RP, PR, SP, PS$$

$$RR, SS, PP$$

where in each pair, the first entry is Player 1's choice. The event $\{RR, SS, PP\}$ occurs with probability

$$(1/4)*(1/4)+(1/2)*(1/4)+(1/4)*(1/2) = 5/16.$$

Therefore, $P(E) = 11/16$. The conditional probability $P(1|E)$ is the ratio

$$\frac{P(RS)+P(SP)+P(PR)}{P(RS)+P(SP)+P(PR)+P(SR)+P(PS)+P(RP)} = 6/11.$$

Therefore, letting p be the probability Player 1 wins, we have the equations:

$$\begin{aligned} P(1) &= P(1|E)P(E)+P(1|E^c)P(E^c) \\ p &= (6/11)(11/16)+p(5/16) \end{aligned}$$

Solving, you get

$$p = 6/11.$$

8 Bayes Method

Problem 8.1: The Vespa auto plant has 3 assembly lines manufacturing cars: Assembly Line 1, Assembly Line 2, Assembly Line 3. Suppose that 50% of new cars are manufactured by Ass. Line 1, and that 35% are manufactured by Ass. Line 2. Suppose that 5% of the cars produced by Ass. Line 1 are defective, that 10% of the cars produced by Ass. Line 2 are defective, and that 15% of the cars produced by Ass. Line 3 are defective.

(a) A car produced by the Vespa auto plant is selected at random. Compute the probability that it is defective.

(b) Given that a car produced by the Vespa auto plant is defective, find out which assembly line most likely produced it.

Solution. Let D,ND be the events that the car is defective, nondefective, respectively. Let A_1, A_2, A_3 be the events that the car was produced by assembly lines 1,2,3, respectively. Start with the array of "forward conditional probabilities":

	D	ND
A_1	0.05	0.95
A_2	0.10	0.90
A_3	0.15	0.85

Multiply row 1 by $P(A_1) = 0.50$, multiply row 2 by $P(A_2) = 0.35$, and multiply row 3 by $P(A_3) = 0.15$:

	D	ND
A_1	0.025	0.475
A_2	0.035	0.315
A_3	0.0225	0.1275

The answer to part(a) is the sum of column 1, which is 0.0825. We obtain the "backward conditional probabilities" by dividing each column of the preceding array by its column sum:

	D	ND
A_1	0.303	0.5177
A_2	0.424	0.3433
A_3	0.273	0.1390

The middle entry in the first column is the biggest. Therefore, assembly line 2 most likely produced the defective car.

Problem 8.2: Consider the following discrete communication system:

$$\boxed{\begin{array}{c}\text{discrete}\\ \text{source}\end{array}} \to X \to \boxed{\begin{array}{c}\text{discrete}\\ \text{channel}\end{array}} \to Y$$

The source output (and channel input) is modeled as a random variable X taking the three possible values $1, 2, 3$ with the probabilities

$$P(X = 1) = 0.4, \quad P(X = 2) = 0.5, \quad P(X = 3) = 0.1.$$

The channel output Y is also modeled as a random variable taking the three possible values $1, 2, 3$. Via observation of many input/output pairs for the channel, it is determined that the following "forward conditional probabilities" give a pretty accurate description of how the channel operates:

$$P(Y = 1|X = 1) = 0.3, \quad P(Y = 2|X = 1) = 0.3, \quad P(Y = 3|X = 1) = 0.4$$

$$P(Y = 1|X = 2) = 0.2, \quad P(Y = 2|X = 2) = 0.4, \quad P(Y = 3|X = 2) = 0.4$$

$$P(Y = 1|X = 3) = 0.4, \quad P(Y = 2|X = 3) = 0.2, \quad P(Y = 3|X = 3) = 0.4$$

(a) Compute $P(Y = 1), P(Y = 2), P(Y = 3)$.

(b) Compute all "backward conditional probabilities" of the form $P(Y = y|X = x)$. (There are 9 such probabilities.)

Solution. Putting the forward conditional probabilities above in a 3×3 matrix, one obtains

$$\begin{array}{ccc} 0.3 & 0.3 & 0.4 \\ 0.2 & 0.4 & 0.4 \\ 0.4 & 0.2 & 0.4 \end{array}$$

Multiply row i by $P(X = i)$, $i = 1, 2, 3$, obtaining the array:

$$\begin{array}{ccc} 0.12 & 0.12 & 0.16 \\ 0.10 & 0.20 & 0.20 \\ 0.04 & 0.02 & 0.04 \end{array} \tag{4}$$

The desired Y probabilities are then the column sums:

$$P(Y = 1) = 0.26, \quad P(Y = 2) = 0.34, \quad P(Y = 3) = 0.40.$$

To obtain the backward conditional probabilities, divide each column of the array (4) by the sum of that column, obtaining the array:

$$\begin{array}{ccc} 12/26 & 12/34 & 16/40 \\ 10/26 & 20/34 & 20/40 \\ 4/26 & 2/34 & 4/40 \end{array}$$

These are the backward conditional probabilities

$$\begin{array}{ccc} P(X = 1|Y = 1) & P(X = 1|Y = 2) & P(X = 1|Y = 3) \\ P(X = 2|Y = 1) & P(X = 2|Y = 2) & P(X = 2|Y = 3) \\ P(X = 3|Y = 1) & P(X = 3|Y = 2) & P(X = 3|Y = 3) \end{array}$$

Problem 8.3: Mary Kay lives in LA and makes frequent consulting trips to Washington, DC. 50% of the time her round trip flight is via Airline 1, 30% of the time it is via Airline 2, and 20% of the time via Airline 3. For Airline 1, LA to DC flights are late 30% of the time, and DC to LA flights are late 10% of the time. For Airline 2, LA to DC flights are late 25% of the time, and DC to LA flights are late 20% of the time. For Airline 3, LA to DC flights are late 40% of the time, and DC to LA flights are late 25% of the time.

(a) On her next consulting trip, what is the probability that Mary Kay will have a late flight both from LA to DC and from DC to LA?

Solution. Take the sample space to consist of the 12 outcomes of the form

(1 or 2 or 3,L or NL, L or NL)

(First slot = airline chosen; second slot tells whether LA-DC flight L or NL; third slot tells whether DC-LA flight L or NL.) The probabilities of the 12 outcomes are (using a tree):

$$\begin{aligned} P(1, L, L) &= (0.50)(0.30)(0.10) = 0.015 \\ P(1, L, NL) &= (0.50)(0.30)(0.90) = 0.135 \\ P(1, NL, L) &= (0.50)(0.70)(0.10) = 0.035 \\ P(1, NL, NL) &= (0.50)(0.70)(0.90) = 0.315 \end{aligned}$$

$$
\begin{aligned}
P(2, L, L) &= (0.30)(0.25)(0.20) = 0.015 \\
P(2, L, NL) &= (0.30)(0.25)(0.80) = 0.06 \\
P(2, NL, L) &= (0.30)(0.75)(0.20) = 0.045 \\
P(2, NL, NL) &= (0.30)(0.75)(0.80) = 0.18 \\
P(3, L, L) &= (0.20)(0.40)(0.25) = 0.02 \\
P(3, L, NL) &= (0.20)(0.40)(0.75) = 0.06 \\
P(3, NL, L) &= (0.20)(0.60)(0.25) = 0.03 \\
P(3, NL, NL) &= (0.20)(0.60)(0.75) = 0.09
\end{aligned}
$$

We want

$$P(1, L, L) + P(2, L, L) + P(3, L, L) = 0.015 + 0.015 + 0.02 = 0.05.$$

(b) On her next consulting trip, what is the probability that Mary Kay will not have a late flight (that is, the LA to DC flight will not be late and the DC to LA flight will not be late)?

Solution. We want

$$P(1, NL, NL) + P(2, NL, NL) + P(3, NL, NL) = 0.315 + 0.18 + 0.09 = 0.585.$$

(c) Mary Kay makes her consulting trip, and then reveals to a friend that exactly one of her two flights was late. Based upon this information, the friend must decide which of the three airlines it was most likely for Mary Kay to have used on her trip. Which airline is this? Which airline would the friend decide was least likely to have been used by Mary Kay? Justify your answers using conditional probabilities.

Solution. Let

$$E = \{(1, L, NL), (1, NL, L), (2, L, NL), (2, NL, L), (3, L, NL), (3, NL, L)\}.$$

We want to see which of the numbers $P(1|E), P(2|E), P(3|E)$ is the biggest and which is the smallest.

$$P(1|E) = (P(1, L, NL) + P(1, NL, L))/P(E) = 34/73.$$

$$P(2|E) = (P(2, L, NL) + P(2, NL, L))/P(E) = 21/73.$$

$$P(3|E) = (P(3, L, NL) + P(3, NL, L))/P(E) = 18/73.$$

Mary is most likely to have taken Airline 1 and least likely to have taken Airline 3.