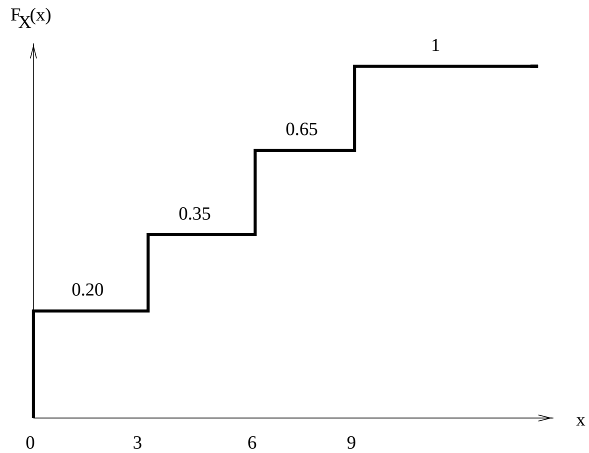
EE 3025 Dr. Kieffer

Chapters 2 and 3 Study Problems

1 CDF's

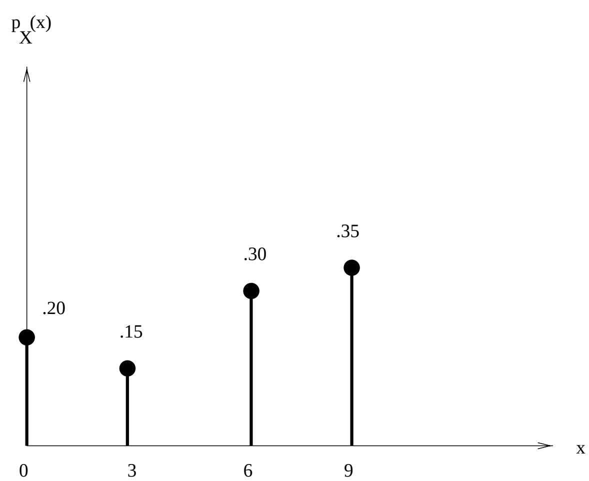
Problem 2.1: A discrete random variable X has the following CDF function $F_X(x)$:



(a) Plot the PMF function $p_X(x)$

(b) Compute the probabilities $P[3 \leq X < 9]$ and $P[3 < X \leq 9]$.

Solution to (a).



Solution to (b).

$$
\begin{aligned}
P[3 \le X < 9] &= p_X(3) + p_X(6) = .45 \\
P[3 < X \le 9] &= p_X(6) + p_X(9) = .65
\end{aligned}
$$

Problem 2.2: A RV X has CDF

$$F_X(x) = (1 - e^{-x}/2)u(x).$$

Find the PDF.

Solution. By the product rule for differentiation,

$$
\begin{aligned}
f_X(x) &= dF_X(x)/dx = (1/2)e^{-x}u(x) + (1 - e^{-x}/2)du(x)/dx \\
&= (1/2)e^{-x}u(x) + (1 - e^{-x}/2)\delta(x) \\
&= (1/2)e^{-x}u(x) + (1/2)\delta(x)
\end{aligned}
$$

(On the last term, I used the "sifting property" of the delta function:

$$\phi(x)\delta(x) = \phi(0)\delta(x),$$

whenever $\phi(x)$ is a function continuous at $x = 0$.) From the form of the PDF, it is clear that we have a mixed random variable.

Problem 2.3: An integer-valued discrete RV has the following table of CDF values:

$$
\begin{aligned}
F_X(0) &= 0.0002 \\
F_X(1) &= 0.0021 \\
F_X(2) &= 0.0123 \\
F_X(3) &= 0.0464 \\
F_X(4) &= 0.1260 \\
F_X(5) &= 0.2639 \\
F_X(6) &= 0.4478 \\
F_X(7) &= 0.6405 \\
F_X(8) &= 0.8011 \\
F_X(9) &= 0.9081 \\
F_X(10) &= 0.9652 \\
F_X(11) &= 0.9894 \\
F_X(12) &= 0.9975 \\
F_X(13) &= 0.9995 \\
F_X(14) &= 0.9999 \\
F_X(15) &= 1.0000 \\
F_X(16) &= 1.0000
\end{aligned}
$$

(a) Compute $P(X = 5)$.

Solution. In all the solutions, I use the formula

$$P(a \leq X \leq b) = F_X(b) - F_X(a-1), \quad a, b \; integers$$

In this case, we have

$$P(X = 5) = P(5 \leq X \leq 5) = F_X(5) - F_X(4) = 0.1379.$$

(b) Compute $P(3 < X \leq 6)$.

Solution.

$$P(3 < X \leq 6) = P(4 \leq X \leq 6) = F_X(6) - F_X(3) = 0.4014.$$

(c) Compute $P(X \geq 7)$.

Solution.

$$\begin{aligned} P(X \geq 7) &= P(7 \leq X \leq \infty) \\ &= F_X(\infty) - F_X(6) = 1 - F_X(6) = 0.5522. \end{aligned}$$

(d) What is the conditional probability that $X > 6$, given that $X > 3$?

Solution.

$$P(X > 6 | X > 3) = \frac{P(X > 6)}{P(X > 3)} = \frac{1 - F_X(6)}{1 - F_X(3)} = 0.58.$$

2 Common Distributions

Problem 3.1: A continuous random variable X is uniformly distributed in the interval $[A, B]$.

(a) Suppose it is known that $\mu_X = -2$ and $\sigma_X^2 = 12$. Find A and find B.

(b) Write a two line MATLAB program which will simulate $10,000$ values of the random variable X.

Solution to (a).

$$\text{Let } L = \text{length of interval}$$

Then

$$L^2/12 = \sigma^2 = 12 \Rightarrow L = 12$$

from which we easily obtain (since μ is the midpoint)

$$\begin{aligned} A &= \mu_X - L/2 = -8 \\ B &= \mu_X + L/2 = 4 \end{aligned}$$

Solution to (b).

Let U be uniform in $[0, 1]$. The mean and variance are $1/2$ and $1/12$, respectively. Translation does not change the variance. So, $U - 1/2$ has mean 0 and variance $1/12$. Scaling does not change the mean. So, $12(U-1/2)$ has mean 0 and variance 12. Translation does not change the variance. So, $12(U - 1/2) - 2 = 12U - 8$ has mean -2 and variance 12, as desired.

Another way to see this is the following. The factor 12 in $12U - 8$ stretches the interval $[0, 1]$ out into the interval $[0, 12]$. The term -8 in $12U - 8$ translates the interval $[0, 12]$ backwards to the interval $[-8, 4]$.

A third approach is simply to normalize U and X to make means equal to zero and variances equal to one. This yields the equation

$$\frac{X+2}{\sqrt{12}} = \frac{U-1/2}{\sqrt{1/12}}$$

which, when simplified, yields $X = 12U - 8$.

Or, take $X = CU + D$ and choose C, D so that $X = -8$ when $U = 0$ and $X = 4$ when $U = 1$.

The program is:

```
u=rand(1,10000);
x=12*u-8;
```

Problem 3.2: An instructor models the score Y that a randomly chosen student makes on an exam as a Gaussian random variable with $\mu_Y = 55$ and $\sigma_Y = 8$. (For example, the instructor could select μ_Y based on the class average, and could select σ_Y based on the sample standard deviation of the scores of the class.)

(a) The instructor has decided that the middle third of the students will receive a "C" on the exam. What range of scores represents a "C"?

(b) What range of scores would represent a "B" grade, if the instructor has decided that 25% of the students should receive a "B"?

Solution to (a). The random variable $Z = (X - 55)/8$ has the distribution $N(0, 1)$. We have

$$P[55 - C \leq Y \leq 55 + C] = .33$$

and must determine the constant C. In terms of Z, we have

$$\begin{aligned} P[55 - C \leq Y \leq 55 + C] &= P[-C/8 \leq Z \leq C/8] \\ &= \Phi(C/8) - \Phi(-C/8) \\ &= \Phi(C/8) - [1 - \Phi(C/8)] \end{aligned}$$

This gives us

$$\begin{aligned} 2\Phi(C/8) - 1 &= .33 \\ \Phi(C/8) &= .67 \\ C/8 &= \Phi^{-1}(.67) = .44 \\ C &= 8(.44) = 3.52 \end{aligned}$$

Rounding down 3.52 to 3.5, this tells us that the "C" range should be 55 ± 3.5, or from 51.5 to 58.5.

Solution to (b). We know from part (a) that 58.5 is the highest "C". We need to determine a threshold B such that

$$P[58.5 \leq Y \leq B] = .25$$

This translates to the inequality

$$P[(58.5 - 55)/8 \leq Z \leq (B - 55)/8] = .25$$

which tells us that

$$\Phi((B - 55)/8) - \Phi(3.5/8) = .25$$

or

$$B = 55 + 8[\Phi^{-1}(.25 + \Phi(3.5/8))] = 66.1937$$

Roughly speaking, the instructor should grant B's in the range from 58.5 to 66.

Problem 3.3: This problem concerns the Gaussian and binomial distributions.

- **(a)** Let X be Gaussian with mean $\mu = 2$ and variance $\sigma^2 = 100$. Using the table on page 142, compute $P[5 \leq X \leq 13]$.
- **(b)** Let Y be a random variable having the binomial distribution with $n = 5$ and $p = 1/4$. Compute $P[Y \geq 2]$.

Solution to (a). $Z = (X - 2)/10$ is standard Gaussian.

$$P[5 \leq X \leq 13] = P[0.3 \leq Z \leq 1.1] = \Phi(1.1) - \Phi(.3) = 0.8643 - 0.6179 = 0.2464$$

Solution to (b).

$$P[Y \geq 2] = 1 - p_Y(0) - p_Y(1) = 1 - \binom{5}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^5 - \binom{5}{1}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^4 = 376/1024 = 47/128$$

Problem 3.4: Over many workdays, it is noticed that phone calls come into the ECE Dept. central number 625-3300 at an averate rate of 0.46 calls/minute. It is desired to assess the likelihood of certain numbers of calls that can occur on the next workday. Let X be the total number of phone calls that will come into the number 625-3300 between 9:00 am and 9:05 am the next workday. Compute

(a) $P(X = 1)$

(b) $P(X \geq 3)$

(c) $P(2 \leq X \leq 4)$

Solution to (a). X is a Poisson(α) RV, with $\alpha = 5 * (0.46) = 2.3$, the expected number of phone calls in a 5 minute period. For a Poisson RV, we have

$$P(X = x) = \alpha^x e^{-\alpha}/x!, \quad x = 0, 1, 2, 3, \cdots.$$

Therefore,

$$P(X = 1) = [\alpha e^{-\alpha}]_{\alpha=2.3} = 0.2306.$$

So, we have a little bit less than 1 chance in 4 that just one call will come in the next workday.

Solution to (b).

$$\begin{aligned} P(X \geq 3) &= 1 - P(X \leq 2) = 1 - P(X = 0) - P(X = 1) - P(X = 2) \\ &= 1 - [e^{-\alpha}]_{\alpha=2.3} - 0.2306 - [\alpha^2 e^{-\alpha}/2]_{\alpha=2.3} \\ &= 1 - 0.1003 - 0.2306 - 0.2652 = 0.4039 \end{aligned}$$

Solution to (c).

$$\begin{aligned} P(X = 2) &= 0.2652 \\ P(X = 3) &= [\alpha^3 e^{-\alpha}/6]_{\alpha=2.3} = 0.2033 \\ P(X = 4) &= [\alpha^4 e^{-\alpha}/24]_{\alpha=2.3} = 0.1169 \end{aligned}$$

Therefore,

$$P(2 \leq X \leq 4) = P(X = 2) + P(X = 3) + P(X = 4) = 0.5854.$$

Problem 3.5: Let RV X denote the lifetime (in hours) of a randomly chosen lite bulb, and we suppose that X has an exponential distribution with PDF

$$f_X(x) = \begin{cases} \exp(-x/1000)/1000, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Compute the following:

(a) $P(X \geq 1000)$.

(b) $P(X \geq 1500 | X \geq 1000)$.

(c) The probability that exactly 3 out of 5 lite bulbs last at least 1000 hours

Solution to (a). We have

$$P(X \geq 1000) = \int_{1000}^{\infty} f_X(x)dx = [-\exp(-x/1000)]_{x=1000}^{x\to\infty} = \exp(-1) = 0.3679.$$

Solution to (b).

$$P(X \geq 1500|X \geq 1000) = \frac{P(X \geq 1500)}{P(X \geq 1000)} = \exp(-1.5)/\exp(-1) = 0.6065.$$

Solution to (c). This part represents an interesting combination of the exponential and binomial distributions. The 5 lite bulbs are tested, one after the other. These tests constitute independent trials. The total number of bulbs that do well on the tests (i.e., last a certain period of time) must therefore follow a Binomial(n,p) distribution with $n = 5$ and p determined by the exponential distribution. In our case here,

$$p = P(X \geq 1000) = 0.3679.$$

Letting Y be the number of bulbs lasting at least 1000 hours, we have

$$P(Y = 3) = \binom{5}{3}(0.3679)^3(1 - 0.3679)^2 = 0.1989.$$

Problem 3.6: The number of message packets arriving at a server in a given time interval is assumed to be a Poisson RV. Assume that the probability of no arrivals in a one millisecond interval is 0.25.

(a) What is the expected number of packets that arrive in a one millisecond interval?

(b) What is the probability that exactly one message packet arrives in a one millisecond interval?

(c) What is the probability that exactly 2 message packets arrive in a 2 millisecond interval?

Solution to (a). Let α denote the expected number of packets arriving in the one millisecond interval. It is given that

$$e^{-\alpha} = 0.25.$$

Solving for α,

$$\alpha = -\ln(0.25) = 1.3863.$$

Solution to (b). According to the Poisson distribution, this probability would be

$$\alpha * e^{-\alpha} = \alpha/4 = 0.3466.$$

Solution to (c). The expected number of arrivals in a 2 millisecond interval would be

$$2\alpha = 2.7726.$$

The number of arrivals in a 2 millisecond interval can therefore be modeled as a Poisson RV with parameter 2.7726. The probability of exactly 2 arrivals would then be

$$(2.7726)^2 * e^{-2.7726}/2 = 0.2402.$$

3 Expected Value

Problem 4.1: Suppose three events A, B, C satisfy:

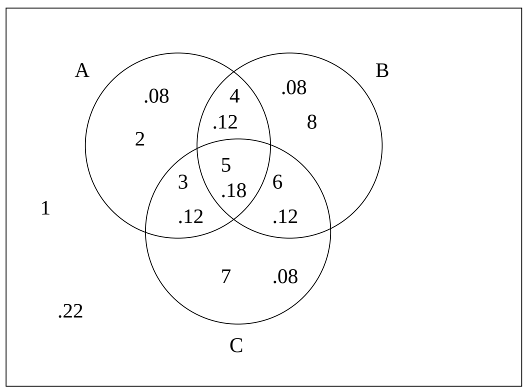
$$P(A) = P(B) = P(C) = 0.5$$

$$P(A \cap B) = P(A \cap C) = P(B \cap C) = 0.3$$

$$P(A \cap B \cap C) = 0.18$$

Let random variable X be the number of events A,B,C which occur on the performance of the experiment. (The possible values of X are obviously $0, 1, 2, 3$.) Determine the expected value of X (the mean of X).

Solution. Using the Venn diagram



we obtain

```
P[X=0] = P(1)= 0.22
P[X=1] = P(2)+P(7)+P(8)= 0.24
P[X=2] = P(3)+P(4)+P(6) = 0.36
P[X=3] = 0.18
E[X] = 0*(0.22) + 1*(0.24) + 2*(0.36) + 3*(0.18) = 1.5
```

Problem 4.2: A discrete random variable X has the PMF

$$p_X(x) = \begin{cases} 0.1, & x = 1 \\ 0.2, & x = 2 \\ 0.3, & x = 3 \\ 0.4, & x = 4 \end{cases}$$

Compute the variance of X.

Solution.

$$\begin{aligned} E[X] &= [1\ 2\ 3\ 4] \bullet [.1\ .2\ .3\ .4] = 3 \\ E[X^2] &= [1\ 4\ 9\ 16] \bullet [.1\ .2\ .3\ .4] = 10 \\ \text{Var}[X] &= E[X^2] - (E[X])^2 = 1 \end{aligned}$$

Problem 4.3: Let the continuous random variable X be uniformly distributed in the interval $[0, 1]$. Let Y be the random variable $Y = \sqrt[3]{X}$.

(a) Find the median of Y.

(b) Find the mean of Y.

Solution to (a). The median of Y is the constant C such that

$$P[Y < C] = 1.$$

Since

$$P[Y < C] = P[X^{1/3} < C] = P[X < C^3] = 1/2,$$

C^3 must be halfway between 0 and 1. This yields $C^3 = 1/2$, and therefore $C = 1/\sqrt[3]{2}$.

Solution to (b).

$$E[Y] = E[X^{1/3}] = \int_0^1 x^{1/3} f_X(x)dx = \int_0^1 x^{1/3}dx = 3/4.$$

Problem 4.4: The discrete random variable X takes the values $-3, -2, -1, 0, 1, 2, 3$ with equal probability. Compute the mean and variance of the random variable $Y = |X| + X$.

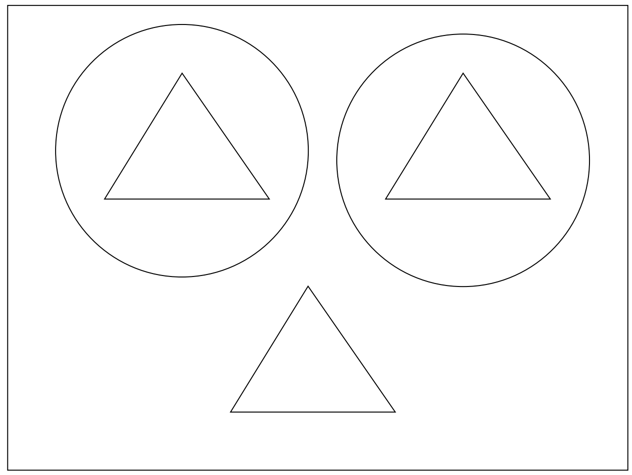
Solution.

$$E[Y] = E[|X|+X] = (1/7)[(3-3)+(2-2)+(1-1)+(0+0)+(1+1)+(2+2)+(3+3) = 12/7.$$

$$E[Y^2] = E[(|X| + X)^2] = (1/7)(2^2 + 4^2 + 6^2) = 8.$$

$$Var[Y] = E[Y^2] - \mu_Y^2 = 8 - (12/7)^2 = 5.06.$$

Problem 4.5: A target filling up the entire back of a state fair booth looks like the following:



A player gets to throw one ball at the target and is guaranteed to hit the target (because in worst case it will bounce off the floor or walls and hit the target the first time, which is the time that counts). At the point where the ball hits the target, a sensor activates, awarding a certain number of tickets to the player (who can accumulate tickets to obtain a kewpie doll).

- 100 tickets are awarded if the ball bits inside a triangle;
- 50 tickets are awarded if the ball hits inside a circle but outside a triangle;
- 20 tickets are awarded if the ball hits anywhere else in the target.

Assume that the player is not a very good player and is therefore equally likely to hit anywhere on the target. Assume that the rectangle is 8 feet by 6 feet, that the triangles are equilateral 2 feet on each side. and that the circles are of diameter 3 feet. Compute the expected number of tickets that the player will win upon throwing the ball just one time.

Solution. Let X be the number of tickets the player wins (on the throwing of one ball). Each of the 3 triangles has area $\sqrt{3}$, so the area of the 100 ticket region is $3\sqrt{3}$. Dividing this by the total area of 48, we get

$$p_X(100) = \frac{3\sqrt{3}}{48} = 0.10825$$

Each of the two circles has area 2.25π and contains a triangle of area $\sqrt{3}$. Therefore, the total area of the 50 ticket region is $2(2.25\pi - \sqrt{3})$. This gives us

$$p_X(50) = \frac{2(2.25\pi - \sqrt{3})}{48} = 0.22236.$$

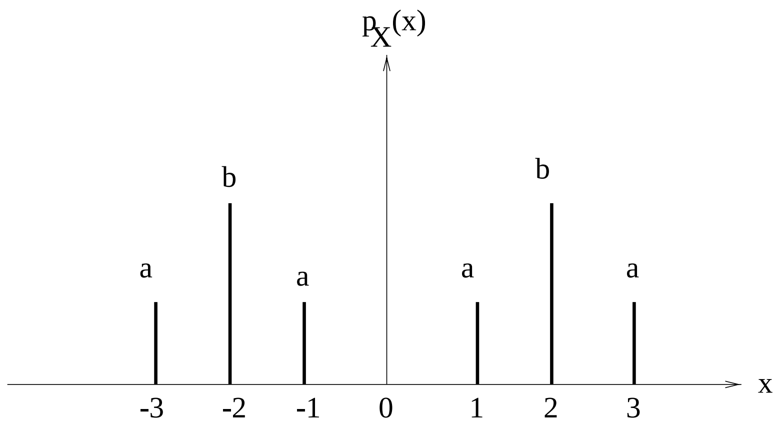
There is only one probability left:

$$p_X(20) = 1 - p_X(100) - p_X(50) = 0.66939.$$

We conclude that

$$E(X) = 100 * (0.10825) + 50 * (0.22236) + 20 * (0.66939) = 35.33.$$

Problem 4.6: A discrete random variable X has values $\pm 3, \pm 2, \pm 1$. Its PMF looks like



The variance of X is 13/3. Find a and b.

Solution. The PMF probabilities must add up to 1. This yields the equation

$$4a + 2b = 1$$

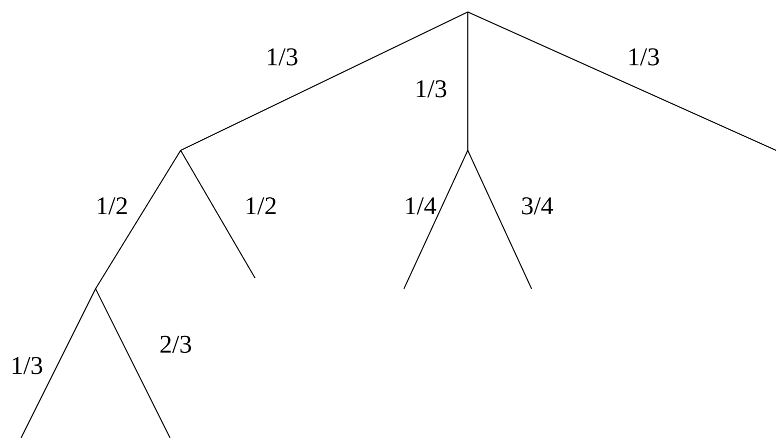
The mean is zero (symmetry). Therefore

$$E[X^2] = 13/3 = 2[a + 4b + 9a] = 2[10a + 4b]$$

Solving this equation simultaneously with the previous equation yields

$$\begin{aligned} a &= 1/12 \\ b &= 1/3 \end{aligned}$$

Problem 4.7: A multiple-step experiment is represented by the following tree:



Find the expected number of steps of the experiment.

Solution. The probability there is one step is 1/3. The probability there is two steps is

$$(1/3)(1/4) + (1/3)(3/4) + (1/3)(1/2) = 1/2$$

The probability there is three steps is

$$1 - (1/3) - (1/2) = 1/6$$

So, the expected number of steps is

$$1 * (1/3) + 2 * (1/2) + 3 * (1/6) = 11/6$$

4 Conditional Distributions

Problem 5.1: A fair coin is flipped. A random variable Y is determined on the basis of this coin flip as follows.

- If the coin is heads, then the conditional density of Y is $y/8$ for $0 \leq y \leq 4$ (zero elsewhere).
- If the coin is tails, then $Y = 6$.

Compute the following:

(a) $E[Y|0 \leq Y \leq 4], E[Y^2|0 \leq Y \leq 4]$.

(b) $E[Y], E[Y^2]$.

(c) $Var[Y]$.

(d) What is the density of Y?

Solution to (a). The conditional density of Y given $0 \leq Y \leq 4$ is the density given when the coin is heads. Therefore,

$$E[Y|0 \leq Y \leq 4] = \int_0^4 y(y/8)dy = 8/3$$

$$E[Y^2|0 \leq Y \leq 4] = \int_0^4 y^2(y/8)dy = 8$$

Solution to (b). Either event $\{Y = 6\}$ occurs (coin is tails), or event $\{0 \leq Y \leq 4\}$ occurs (coin is heads), each event occuring with prob 1/2. Therefore,

$$E[Y] = (1/2)E[Y|0 \leq Y \leq 4] + (1/2)E[Y|Y = 6] = (1/2)(8/3) + (1/2)6 = 13/3$$

$$E[Y^2] = (1/2)E[Y^2|0 \leq Y \leq 4] + (1/2)E[Y^2|Y = 6] = (1/2)(8) + (1/2)36 = 22$$

Solution to (c).

$$Var[Y] = E[Y^2] - \mu_Y^2 = 22 - (13/3)^2 = 3.22.$$

Solution to (d). Take 1/2 times the cond density of Y when the coin is heads plus 1/2 times the cond density of Y when the coin is tails. This gives

$$f_Y(y) = 1/2[(y/8)\{u(y) - u(y-8)\}] + 1/2[\delta(y-6)],$$

meaning that Y has a mixed distribution. (We could have computed the mean and variance of Y directly from this density instead of the method used in parts (a)-(c), but that would have been a bit messier.)

Problem 5.2: A random variable X has the PDF

$$f_X(x) = \begin{cases} 1/4, & 0 \leq x \leq 2 \\ 1/2, & 2 < x \leq 3 \\ 0, & \text{elsewhere} \end{cases}$$

Use conditional distributions to do all the following computations.

(a) Compute $P[0 \leq X \leq 1|0 \leq X \leq 2]$.

(b) Compute $E[X|0 \leq X \leq 2]$.

(c) Compute $E[X|2 < X \leq 3]$.

(d) Compute $E[X]$.

Solution to (a). Given $0 \leq X \leq 2$, X is conditionally uniformly distributed on that interval. Automatically, we can therefore say that

$$P[0 \leq X \leq 1|0 \leq X \leq 2] = 1/2.$$

Solution to (b) and (c). Given $0 \le X \le 2$, the cond mean must be the midpoint of interval [0,2] (since this cond dist is uniform). Therefore,

$$E[X|0 \le X \le 2] = 1.$$

Similarly, we automatically have

$$E[X|2 < X \le 3] = 2.5.$$

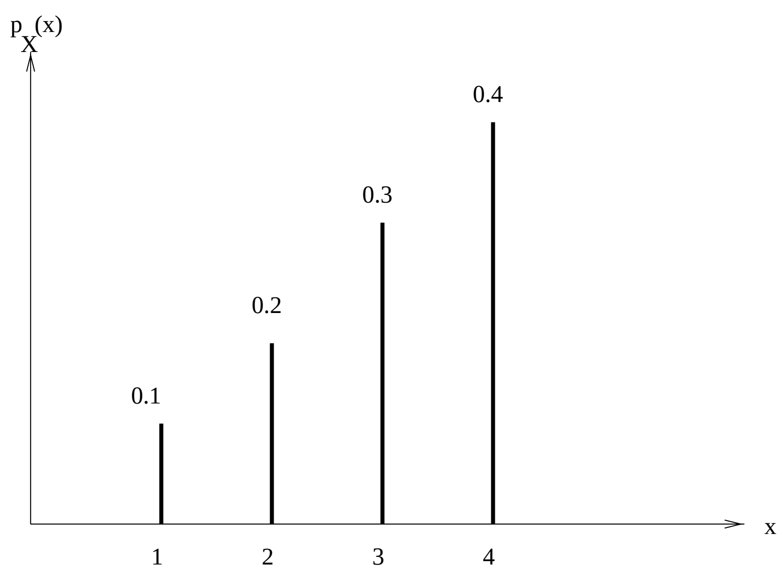
Solution to (d).

$$E[X] = P[0 \le X \le 2]E[X|0 \le X \le 2] + P[2 < X \le 3]E[X|2 < X \le 3].$$

Plugging in, we get

$$(1/2) * 1 + (1/2) * 2.5 = 1.75.$$

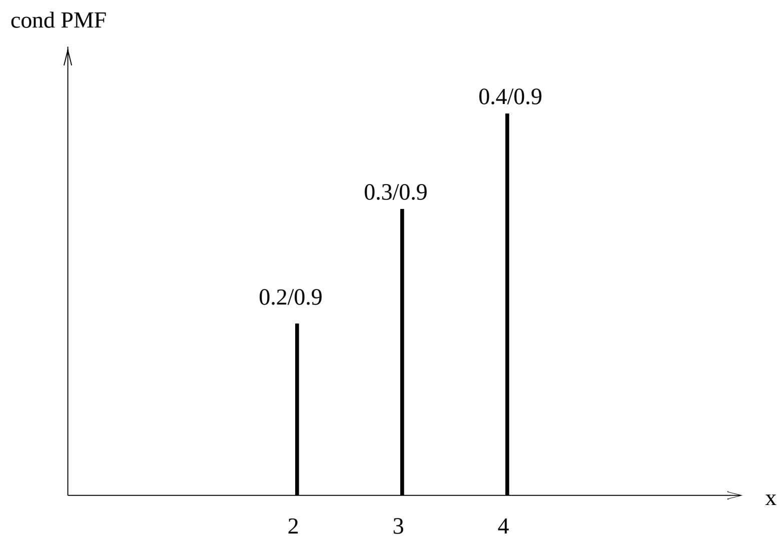
Problem 5.3: A discrete random variable X has the following PMF function $p_X(x)$:

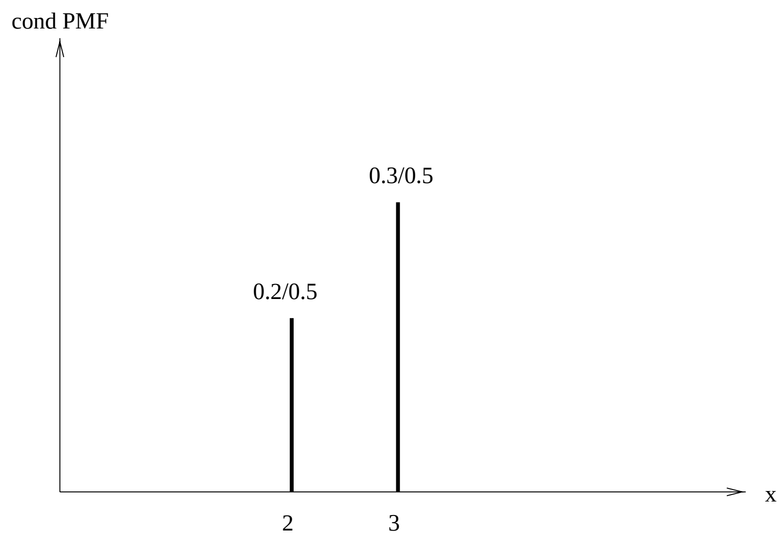


(a) Let B_1 be the event $B_1 = \{X \ge 2\}$. Plot the conditional PMF $p_{X|B_1}(x)$ of X given B_1.

(b) Let B_2 be the event $B_2 = \{2 \le X \le 3\}$. Plot the conditional PMF $p_{X|B_2}(x)$ of X given B_2.

Solution to (a). Remove the values of X not satisfying the condition, and then re-normalize the remaining probabilities so that they sum up to one.





Solution to (b).

Problem 5.4: Let X be a Gaussian random variable with $\mu_X = 1$ and $\sigma_X^2 = 4$. Let Z be the random variable $Z = (X - \mu_X)/\sigma_X$. Compute the conditional expected value

$$E[Z|0 \leq X \leq 1]$$

using the Table on page 142.

Solution. Since $Z = (X-1)/2$, we see that $\{0 \leq X \leq 1\} = \{-1/2 \leq Z \leq 0\}$. This gives us

$$E[Z|0 \leq X \leq 1] \;\; = \;\; E[Z| - 0.5 \leq Z \leq 0]$$

$$
\begin{aligned}
&= \frac{\int_{-0.5}^{0} z f_Z(z)dz}{P[-0.5 \le Z \le 0]} \\
&= \frac{\int_{-0.5}^{0} (1/\sqrt{2\pi}) z \exp(-z^2/2)dz}{\Phi(0) - (1 - \Phi(0.5))} \\
&= \frac{[-(1/\sqrt{2\pi}) \exp(-z^2/2)]_{z=-0.5}^{z=0}}{0.5 - (1 - 0.6915)} \\
&= \frac{(1/\sqrt{2\pi})[-1 + \exp(-1/8)]}{0.1915} \\
&\approx -0.245
\end{aligned}
$$

Problem 5.5: A random variable X has the density $f_X(x) = e^{-x}u(x)$. Compute

- **(a)** $P[1 < X < 2|X < 4]$
- **(b)** $E[X|X < 4]$
- **(c)** $Var[X|X < 4]$

Solution of (a). Letting B be the event $B = \{X < 4\}$, we have

$$
P[1 < X < 2|X < 4] = \int_1^2 f_{X|B}(x)dx \tag{1}
$$

Substituting

$$
f_{X|B}(x) = \begin{cases} f_X(x)/P[B], & 0 < x < 4 \\ 0, & \text{elsewhere} \end{cases}
$$

in (1), we have

$$
\begin{aligned}
P[1 < X < 2|X < 4] &= \frac{\int_1^2 e^{-x}dx}{\int_0^4 e^{-x}dx} \\
&= \frac{e^{-1} - e^{-2}}{1 - e^{-4}} = .2329
\end{aligned}
$$

Solution of (b).

$$
\begin{aligned}
E[X|X < 4] &= \int x f_{X|B}(x)dx \\
&= \frac{\int_0^4 x e^{-x}dx}{\int_0^4 e^{-x}dx} \\
&= \frac{1 - 5e^{-4}}{1 - e^{-1}} = 0.9254
\end{aligned}
$$

Solution of (c).

$$E[X^2|X<4] = \frac{\int_0^4 x^2 e^{-x} dx}{\int_0^4 e^{-x} dx}$$
$$Var[X|X<4] = E[X^2|X<4] - (E[X|X<4])^2$$

The reader is invited to finish the above calculation himself/herself.

5 Moment Generating Functions

Problem 6.1: The moment generating function for a RV X is

$$M_X(s) = \frac{8}{(2-s)^3}.$$

Compute μ_X and σ_X^2.

Solution. The first derivative is

$$\frac{24}{(2-s)^4}.$$

Evaluating this at $s=0$, we get the mean:

$$\mu_X = 3/2.$$

The second derivative is

$$\frac{96}{(2-s)^5}.$$

Evaluating at $s=0$, we get $E(X^2)$:

$$E(X^2) = 3.$$

Therefore,

$$\sigma_X^2 = E(X^2) - \mu_X^2 = 3 - 9/4 = 3/4.$$

Problem 6.2: A random variable X has MGF

$$M_X(s) = (1/3)e^{-3s} + (2/3)e^{2s}.$$

Compute the mean and variance of X.

Solution. The first derivative of the MGF is

$$-e^{-3s} + (4/3)e^{2s}.$$

Plugging in $s = 0$, we see that the mean is

$$\mu_X = -1 + (4/3) = 1/3.$$

The second derivative of the MGF is

$$3e^{-3s} + (8/3)e^{2s}.$$

Plugging in $s = 0$, we obtain the second moment:

$$E[X^2] = 3 + (8/3) = 17/3.$$

We can now compute $Var[X]$:

$$Var[X] = E[X^2] - \mu_X^2 = 17/3 - (1/3)^2 = 50/9.$$

Problem 6.3: A binomial random variable X has MGF

$$M_X(s) = (pe^s + 1 - p)^n.$$

Show that the mean and variance are np and $np(1-p)$, respectively.

Solution. Differentiating one time, it is easy to see the relationship:

$$(pe^s + 1 - p)M_X'(s) = npe^s M_X(s).$$

Plugging in $s = 0$, and using the fact that $M_X(0) = 1$:

$$\mu_X = M_X'(0) = np$$

Differentiating again,

$$(pe^s + 1 - p)M_X''(s) + pe^s M_X'(s) = npe^s(M_X(s) + M_X'(s)),$$

and so

$$M_X''(0) + pM_X'(0) = np(1 + M_X'(0)).$$

Putting in $M_X'(0) = np$, this simplifies to

$$E[X^2] = M_X''(0) = np(1 + np) - np^2.$$

Subtracting off $(np)^2$, the square of the mean, you get

$$Var[X] = np - np^2 = np(1-p).$$

Problem 6.4: A Poisson random variable X has MGF

$$M_X(s) = \exp(\alpha e^s - 1).$$

Show that the mean and variance are both α.

Solution. The first derivative yields

$$M_X'(s) = \alpha e^s M_X(s).$$

Plugging in $s = 0$ immediately yields

$$\mu_X = M_X'(0) = \alpha M_X(0) = \alpha.$$

Differentiating one more time:

$$M_X''(s) = \alpha e^s (M_X(s) + M_X'(s)).$$

Then:

$$E[X^2] = M_X''(0) = \alpha(1 + M_X'(0)) = \alpha(1 + \alpha).$$

Subtracting α^2, the square of the mean, we get

$$Var[X] = \alpha,$$

too.

Problem 6.5: A Gaussian random variable X has MGF

$$M_X(s) = \exp(\mu s + \sigma^2 s^2/2).$$

Show that the mean and variance are respectively equal to μ, σ^2.

Solution.

$$M_X'(s) = (\mu + \sigma^2 s)M_X(s),$$

and then

$$\mu_X = M_X'(0) = \mu M_X(0) = \mu$$

is immediate. Differentiating again:

$$M_X''(s) = \sigma^2 M_X(s) + (\mu + \sigma^2 s)M_X'(s).$$

$$E[X^2] = M_X''(0) = \sigma^2 M_X(0) + \mu M_X'(0) = \sigma^2 + \mu^2.$$

Subtracting off μ^2, the square of the mean, we obtain

$$Var[X] = (\sigma^2 + \mu^2) - \mu^2 = \sigma^2.$$