EE 3025 Dr. Kieffer

Chapters 4 and 5 Solved Problems

1 Distribution of Function of One RV

Problem 1.1: Let U be a random variable uniformly distributed between 0 and 1.

(a) Find a function $z = \phi(u)$ such that the random variable $Z = \phi(U)$ will have the PDF

$$f_Z(z) = \begin{cases} 0.4(z+1), & 1 \le z \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

(b) Write a two line MATLAB program which will simulate $10,000$ samples of the random variable Z in (a).

Solution to (a).
The CDF satisfies

$$F_Z(z) = \int_1^z 0.4(z+1)dz = 0.2(z+1)^2 - 0.8, \quad 1 \le z \le 2.$$

Set

$$U = 0.2(Z+1)^2 - 0.8,$$

and solve for Z in terms of U:

$$Z = -1 + \sqrt{5U+4}.$$

Solution to (b).

```
u=rand(1,10000);
z=-1+sqrt(5*u+4);
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Problem 1.2: $X \sim Gaussian(0,1)$ be the input to a hard limiter. The output is the random variable Y in which

$$Y = \begin{cases} C, & X \ge C \\ X, & -C < X < C \\ -C, & X \le -C \end{cases}$$

(C is an unspecified positive constant.) Find the PDF of Y.

Solution. Notice that Y is mixed: it takes the discrete values $\pm C$, and it is also continuously distributed over the interval from $-C$ to C. The PDF therefore takes the form

$$f_Y(y) = p_Y(C)\delta(y - C) + p_Y(-C)\delta(y + C) + g(y)$$

where $g(y)$ is a function taking finite values that vanishes everywhere except between $-C$ and C. Notice that

$$\begin{aligned} p_Y(C) &= P[Y = C] = P[X \geq C] = 1 - \Phi(C) \\ p_Y(-C) &= P[Y = -C] = P[X \leq -C] = \Phi(-C) = p_Y(C) \end{aligned}$$

Fix an arbitrary y satisfying $-C < y < C$. Then

$$P[-C < Y < y] = \int_{-C}^{y} f_Y(y)dy = \int_{-C}^{y} g(y)$$

Differentiating both sides with respect to y, we get

$$(d/dy)P[-C < Y < y] = g(y)$$

Also, we have

$$P[-C < Y < y] = P[-C < X < y] = \int_{-C}^{y} f_X(x)dx$$

Differentiating both sides with respect to y, we obtain

$$(d/dy)P[-C < Y < y] = f_X(y) = \frac{1}{\sqrt{2\pi}} \exp(-y^2/2)$$

We conclude that

$$g(y) = \begin{cases} 0, & y \leq -C \\ \frac{1}{\sqrt{2\pi}} \exp(-y^2/2), & -C < y < C \\ 0, & y \geq C \end{cases}$$

2 Joint PMF's and Joint PDF's

Problem 2.1: An urn contains three cards numbered "1", four cards numbered "2", and five cards numbered "3". Two cards are selected at random. Let X be the number on the first card selected and let Y be the number on the second card selected.

(a) Let $p(x, y)$ denote the joint PMF. Find the joint probability matrix $[p(x, y)]$ under sampling without replacement and sampling with replacement.

(b) Compute $P[X = Y], P[X > Y], P[Y > X]$ under sampling without replacement and sampling with replacement.

Solution to (a). Sampling with or without replacement, we have

$$[p_X(1) \;\; p_X(2) \;\; p_X(3)] = [3/12 \;\; 4/12 \;\; 5/12]$$

Sampling without replacement, we have

$$[p(y|x)] = \begin{bmatrix} 2/11 & 4/11 & 5/11 \\ 3/11 & 3/11 & 5/11 \\ 3/11 & 4/11 & 4/11 \end{bmatrix}$$

from which we obtain

$$[p(x,y)] = \begin{bmatrix} 3/12 & 0 & 0 \\ 0 & 4/12 & 0 \\ 0 & 0 & 5/12 \end{bmatrix} [p(y|x)] = \begin{bmatrix} 6/132 & 12/132 & 15/132 \\ 12/132 & 12/132 & 20/132 \\ 15/132 & 20/132 & 20/132 \end{bmatrix}$$

Sampling with replacement, we have

$$[p(y|x)] = \begin{bmatrix} 3/12 & 4/12 & 5/12 \\ 3/12 & 4/12 & 5/12 \\ 3/12 & 4/12 & 5/12 \end{bmatrix}$$

and therefore

$$[p(x,y)] = \begin{bmatrix} 3/12 & 0 & 0 \\ 0 & 4/12 & 0 \\ 0 & 0 & 5/12 \end{bmatrix} [p(y|x)] = \begin{bmatrix} 9/144 & 12/144 & 15/144 \\ 12/144 & 16/1444 & 20/144 \\ 15/144 & 20/144 & 25/144 \end{bmatrix}$$

Solution to (b). For sampling without replacement,

$$P[X = Y] = P_{X,Y}(1,1) + P_{X,Y}(2,2) + P_{X,Y}(3,3) = 38/132$$

The $[p(x,y)]$ matrix is symmetric, and so $P[X > Y]$ and $P[Y > X]$ are the same. Since

$$P[X = Y] + P[X > Y] + P[Y < X] = 1$$

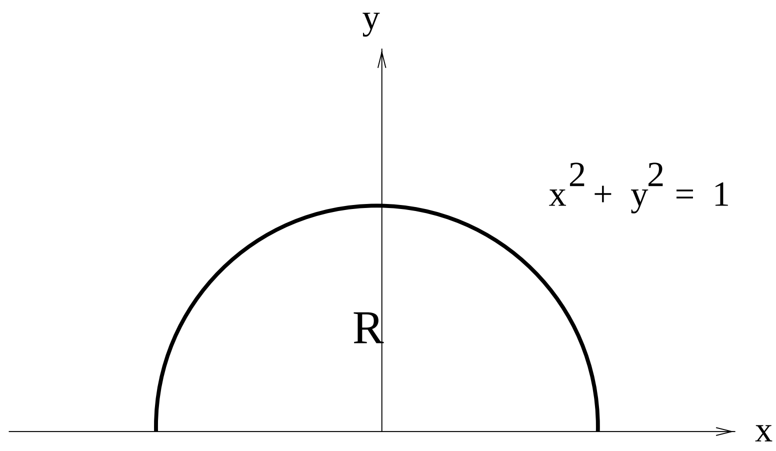
we conclude that

$$P[X > Y] = P[Y > X] = (1/2)[1 - P[X = Y]] = 47/132$$

For sampling with replacement, the matrix $[p(x,y)]$ is also symmetric and we have by the same arguments that were applied to sampling without replacement that:

$$\begin{aligned} P[X = Y] &= P_{X,Y}(1,1) + P_{X,Y}(2,2) + P_{X,Y}(3,3) = 50/144 \\ P[X > Y] &= P[Y > X] = (1/2)[1 - P[X = Y]] = 47/144 \end{aligned}$$

We can make some further remarks. In both sampling with replacement and sampling without replacement, we can say that X and Y have the same PMF (which follows because $[p(x,y)]$ is a symmetric matrix in each case). However, the situation is quite different in terms of how X and Y relate to each other. In sampling without replacement, the random variables X and Y are statistically dependent, whereas under sampling with replacement the random variables are statistically independent (which means that $p(x,y) = p_X(x)p_Y(y)$).



Problem 2.2: Let R be the following semicircular region plotted above. Suppose (X, Y) has density Cy in the region R (zero elsewhere).

(a) Compute C.

(b) Compute $P[X^2 + Y^2 \geq 0.5]$.

Solution.
(a)

$$\begin{aligned} C &= \frac{1}{\iint_R y dy dx} \\ &= \frac{1}{\int_0^1 \int_0^\pi r^2 \sin\theta d\theta dr} \\ &= 1.5 \end{aligned}$$

(b)
Converting to polar coordinates, the desired probability is computable as:

$$\int_0^\pi \int_{\sqrt{.5}}^1 (1.5) r^2 \sin\theta dr d\theta \approx .65$$

3 Marginal Distributions

Problem 3.1: We are given the following matrix of joint probabilities

$$[p(x,y)] = \begin{bmatrix} .1 & 0 & .2 \\ .05 & .2 & .3 \\ .1 & 0 & .05 \end{bmatrix}$$

We suppose that the values of X are $0, 1, 2$ and the values of Y are also $0, 1, 2$. Find the marginal PMF's $p_X(x)$ and $p_Y(y)$.

Solution. The row sums of $[p(x,y)]$ are $.3, .55, .15$, and so

$$[p_X(0) \;\; p_X(1) \;\; p_X(2)] = [.3 \;\; .55 \;\; .15].$$

The column sums of $[p(x,y)]$ are $.25, .2, .55$, and so

$$[p_Y(0) \;\; p_Y(1) \;\; p_Y(2)] = [.25 \;\; .2 \;\; .55].$$

X \ Y	0	1	2	3
0	C	0	0	C
1	0	C/2	C/2	0
2	0	C/2	C/2	0
3	C	0	0	C

Problem 3.2: Let X, Y be discrete RV's having the joint PMF given by the preceding table. Find the marginal distributions $p_X(x)$ and $p_Y(y)$.

Solution. First, we have to find C. Since all 8 nonzero probabilities must add up to 1, it follows that $C = 1/6$. We can then obtain $p_X(x)$ and $p_Y(y)$ from the row sums and column sums, respectively:

$$p_X(0) = 1/3, \;\; p_X(1) = 1/6, \;\; p_X(2) = 1/6, \;\; p_X(3) = 1/3.$$

$$p_Y(0) = 1/3, \;\; p_Y(1) = 1/6, \;\; p_Y(2) = 1/6, \;\; p_Y(3) = 1/3.$$

Problem 3.3: Given that

$$f_{X,Y}(x,y) = \begin{cases} ax + by, & 0 \le x, y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

and that $E(Y) = 11/18$, find $f_Y(y)$, $f_X(x)$.

Solution. First, we have to find the constants a, b. Two equations involving a, b are:

$$\begin{aligned} \int_0^1 \int_0^1 (ax + by)\,dx\,dy &= 1 \\ \int_0^1 \int_0^1 y(ax + by)\,dx\,dy &= 11/18 \end{aligned}$$

These reduce to:

$$
\begin{aligned}
(a/2 + (b/2) &= 2 \\
(a/4) + (b/3) &= 11/18
\end{aligned}
$$

The solutions are $a = 2/3$ and $b = 4/3$.

$$f_Y(y) = \int_0^1 [(2x + 4y)/3]dx = (1 + 4y)/3, \quad 0 \le y \le 1.$$

(zero elsewhere)

$$f_X(x) = \int_0^1 [(2x + 4y)/3]dy = (2x + 2)/3, \quad 0 \le x \le 1.$$

(zero elsewhere)

4 Independence of Two RV's

Problem 4.1: A pair of discrete random variables X, Y has joint PMF $p(x, y)$ given by

$$
\begin{array}{lll}
p(0,0) = .08 & p(0,1) = .12 & p(0,2) = .20 \\
p(1,0) = .12 & p(1,1) = .18 & p(1,2) = .30
\end{array}
$$

Prove that X, Y are independent RV's.

Solution. First, find the marginal PMF's $p_X(x)$ and $p_Y(y)$. Taking the column and row sums, you get

$$
\begin{aligned}
p_Y &= [.20 \;\; .30 \;\; .50] \\
p_X &= [.40 \;\; .60]
\end{aligned}
$$

Forming the products $p_X(x)p_Y(y)$, you get the matrix

$$
\begin{bmatrix}
(.40)(.20) & (.40)(.30) & (.40)(.50) \\
(.60)(.20) & (.60)(.30) & (.60)(.50)
\end{bmatrix}
$$

which is the same as the matrix we started with. Therefore, we have independence.

Problem 4.2: Are the RV's X, Y in Problem 2.2 independent?

Solution. No. The semicircular region R is not rectangular. (Under independence, the joint density $f_X(x)f_Y(y)$ would be nonvanishing over a rectangular region, not a semicircular region.)

Problem 4.3: Consider the joint density

$$f_{X,Y}(x,y) = \begin{cases} Cxy, & 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

Are X, Y independent?

Solution. You can factor as follows

$$f_{X,Y}(x,y) = C\phi(x)\psi(y),$$

over the entire (x,y) plane, where

$$\phi(x) = \begin{cases} x, & 0 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$\psi(y) = \begin{cases} y, & 0 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

Therefore, X, Y are independent. The PDF $f_X(x)$ must be a multiple of x from 0 to 1. There is only one such density:

$$f_X(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

Similarly,

$$f_Y(y) = \begin{cases} 2y, & 0 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

Since $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ in this case, we must have

$$C = 2 * 2 = 4.$$

Problem 4.4: Consider the joint density

$$f_{X,Y}(x,y) = \begin{cases} Cxy, & 0 \le x \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

Are X, Y independent?

Solution. There is no factorization

$$f_{X,Y}(x,y) = C\phi(x)\psi(y),$$

valid over the entire (x,y) plane. (If there were, $\phi(x)$ would take positive values for $0 < x < 1$ and $\psi(y)$ would take positive values for $0 < y < 1$, and therefore $f_{X,Y}(x,y)$ would take positive values over the entire square

$$0 < x < 1, \quad 0 < y < 1.$$

This is impossible because the region of positivity of $f_{X,Y}(x,y)$ is a triangular, not square.)

We conclude that X, Y are dependent.

Problem 4.5: Consider the joint density

$$f_{X,Y}(x,y) = C * e^{-(x^2+y^2)},$$

over the entire (x,y) plane. Are X,Y independent?

Solution. We have the factorization

$$f_{X,Y}(x,y) = C * e^{-x^2} e^{-y^2},$$

valid over the entire (x,y) plane.

We conclude that X,Y are independent. The PDF $f_X(x)$ must be a constant times e^{-x^2}, and must therefore be the density

$$\frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2},$$

with $2\sigma^2 = 1$, or $\sigma = 1/\sqrt{2}$. The densities of X and Y are therefore:

$$f_X(x) = \frac{1}{\sqrt{\pi}} \exp(-x^2)$$

$$f_Y(y) = \frac{1}{\sqrt{\pi}} \exp(-y^2)$$

The constant C is therefore

$$C = (1/\sqrt{\pi})^2 = 1/\pi.$$

Problem 4.6: Consider the joint density

$$f_{X,Y}(x,y) = \begin{cases} e^{-(x+y)}, & 0 \le x \le 1; \quad 0 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

Are X,Y independent?

Solution. Let

$$\phi(x) = e^{-x} u(x)$$

$$\psi(y) = e^{-y} u(y)$$

The factorization

$$f_{X,Y}(x,y) = \phi(x)\psi(y)$$

holds over the entire (x,y) plane.

We conclude that X,Y are independent. Also, X and Y each have the exponential distribution with parameter 1.

Problem 4.7: Consider the joint density

$$f_{X,Y}(x,y) = \begin{cases} Ce^{-(x+y)}, & 0 \le x \le y < \infty \\ 0, & \text{elsewhere} \end{cases}$$

Are X, Y independent?

Solution. Since the region of positivity of $f_{X,Y}(x,y)$ is triangular and not rectangular, there is no factorization

$$f_{X,Y}(x,y) = \phi(x)\psi(y)$$

over the entire (x, y) plane. We conclude that X, Y are dependent. The reader who does not believe this can calculate the individual densities. For $0 \le x < \infty$,

$$f_X(x) = \int_x^\infty C\exp(-x-y)dy = Ce^{-2x},$$

and therefore $C = 2$. For $0 \le y < \infty$,

$$f_Y(y) = \int_0^y 2\exp(-x-y)dx = 2(e^{-y} - e^{-2y}).$$

The product of these is not $f_{X,Y}(x,y)$.

5 Correlation and Covariance

Problem 5.1: Given that U, V are random variables each having variance equal to 7, and given that $\sigma_{U,V} = -4$, compute the value of the constant C that will make the variables U and $U - CV$ uncorrelated. In other words, solve the following equation for C:

$$\text{Cov}(U, U - CV) = 0$$

Solution.

$$\begin{aligned} \text{Cov}(U, U - CV) &= \text{Cov}(U,U) - C\,\text{Cov}(U,V) \\ &= 7 - C(-4) = 0 \end{aligned}$$

The solution is now clearly seen to be $C = -7/4$. (Notice that the fact that $\text{Var}[V] = 7$ was not used.)

Problem 5.2: X and Y are unspecified random variables for which it is given that

$$\begin{aligned} \sigma_X^2 &= 1 \\ \sigma_Y^2 &= 4 \\ \sigma_{X,Y} &= -1 \end{aligned}$$

(a) Compute $\text{Cov}(2X + 3Y, 4X - 5Y)$

(b) Letting $U = 2X + 3Y$ and letting $V = 4X - 5Y$, compute $\rho_{U,V}$.

Solution to (a).

$$\begin{aligned} \text{Cov}(2X+3Y, 4X-5Y) &= 8\text{Cov}(X,X) + 2\text{Cov}(X,Y) - 15\text{Cov}(Y,Y) \\ &= 8(1) + 2(-1) - 15(4) = -54 \end{aligned}$$

Solution to (b).

$$\begin{aligned} \text{Var}[U] &= \text{Cov}(2X+3Y, 2X+3Y) \\ &= 4\text{Cov}(X,X) + 12\text{Cov}(X,Y) + 9\text{Cov}(Y,Y) \\ &= 4(1) + 12(-1) + 9(4) = 28 \\ \text{Var}[V] &= \text{Cov}(4X-5Y, 4X-5Y) \\ &= 16\text{Cov}(X,X) - 40\text{Cov}(X,Y) + 25\text{Cov}(Y,Y) \\ &= 16(1) + (-40)(-1) + 25(4) = 156 \end{aligned}$$

We conclude that

$$\rho_{U,V} = -54/\sqrt{28}\sqrt{156} = -0.8171$$

Problem 5.3: It is given that RV's X and Y satisfy the following:

$$\begin{aligned} \text{Var}(X) &= 4 \\ \text{Var}(Y) &= 1 \\ \text{Cov}(2X+3Y, 4X-7Y) &= 8 \end{aligned}$$

(a) Find $\text{Cov}(X,Y)$.
Solution.

$$\text{Cov}(2X+3Y, 4X-7Y) = 8\text{Var}(X) - 21\text{Var}(Y) - 2\text{Cov}(X,Y) = 11 - 2\text{Cov}(X,Y) = 8$$

It follows that

$$\text{Cov}(X,Y) = 1.5$$

(b) Find $\rho_{X,Y}$.
Solution.

$$\rho_{X,Y} = \text{Cov}(X,Y)/(\sigma_X \sigma_Y) = 1.5/(2 * 1) = 0.75$$

(c) Find the constant A such that $\text{Var}(Y - AX)$ is as small as it can possibly be.
Solution.

$$\text{Var}(Y - AX) = \text{Var}(Y) + A^2\text{Var}(X) - 2A\text{Cov}(X,Y) = 1 + 4A^2 - 3A$$

Setting the derivative with respect to A equal to zero, you get $A = 3/8$.

Problem 5.4: U, V, W are independent mean 0 variance 1 RV's. Let

$$\begin{aligned} X &= U + V - W \\ Y &= U + W - V \\ Z &= V + W - U \end{aligned}$$

(a) Find the covariance matrix of X, Y, Z.

Solution. Let A be the coefficient matrix of U, V, W:

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

Then the covariance matrix of X, Y, Z is $A \Sigma A^T$, where Σ is the covariance matrix of U, V, W. In this case, Σ is the 3×3 identity matrix and therefore it can be ignored; we need only compute AA^T, which is

$$\begin{bmatrix} \sigma_X^2 & \sigma_{X,Y} & \sigma_{X,Z} \\ \sigma_{X,Y} & \sigma_Y^2 & \sigma_{Y,Z} \\ \sigma_{X,Z} & \sigma_{Y,Z} & \sigma_Z^2 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}.$$

Therefore, all three covariances are equal to -1.

(b) Find $\rho_{X,Y}$, $\rho_{X,Z}$, $\rho_{Y,Z}$.

Solution. These are all equal to $-1/3$.

Problem 5.5: Given that X, Y are independent random variables, each having variance one, compute the variance of the random variable $3X + 2Y$ and compute the variance of the random variable $3X - 2Y$.

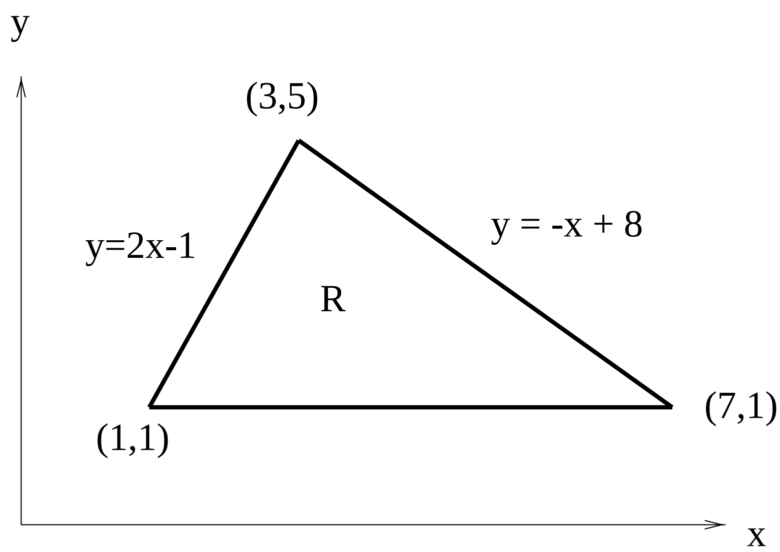
Solution.

$$\begin{aligned} \mathrm{Var}[3X + 2Y] &= \mathrm{Cov}(3X + 2Y, 3X + 2Y) = 9\sigma_X^2 + 12\sigma_{X,Y} + 4\sigma_Y^2 \\ \mathrm{Var}[3X - 2Y] &= \mathrm{Cov}(3X + 2Y, 3X + 2Y) = 9\sigma_X^2 - 12\sigma_{X,Y} + 4\sigma_Y^2 \end{aligned}$$

In each case, the middle term drops out because $\sigma_{X,Y} = 0$ for independent random variables. So

$$\mathrm{Var}[3X + 2Y] = \mathrm{Var}[3X - 2Y] = 13$$

6 Center of Gravity Problems



Problem 6.1: We consider random variables X, Y jointly uniformly distributed in the triangular region R sketched above. Find $E[X]$ and $E[Y]$.

Solution. The point $(E[X], E[Y])$ is the centroid of the triangle, which is easily computed as

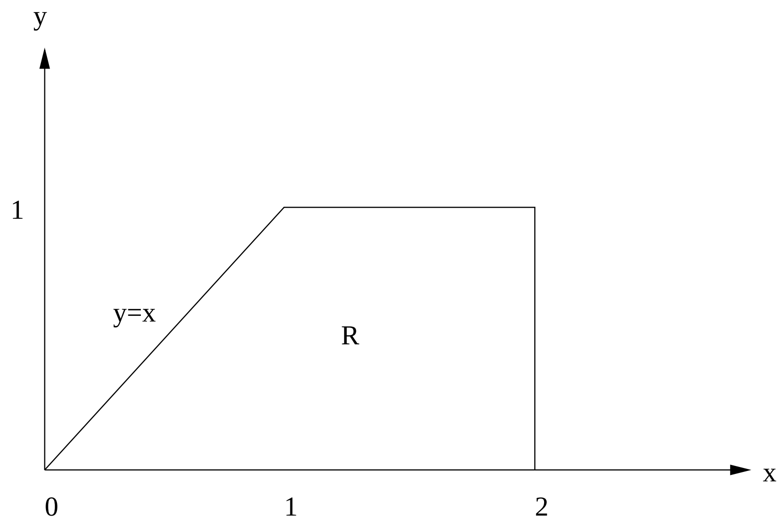
$$(1/3)[(3,5) + (1,1) + (7,1)] = (11/3, 7/3)$$

Or, the centroid is two-thirds of the way down the median connecting the points $(3,5)$ and $(4,1)$:

$$(E[X], E[Y]) = (1/3)(3,5) + (2/3)(4,1) = (11/3, 7/3)$$

Therefore

$$\begin{aligned} E[X] &= 11/3 \\ E[Y] &= 7/3 \end{aligned}$$



Problem 6.2: (X, Y) is uniform over the preceding trapezoidal region R. Find $E[X]$ and $E[Y]$.

Solution. $(E(X), E(Y))$ is the geometric centroid of R. The centroid of the square part of R formed by the 4 vertices $(1,0),(2,0),(1,1),(2,1)$ is $(1.5, 0.5)$. The centroid of the triangular part of R formed by the 3 vertices $(0,0),(1,0),(1,1)$ is

$$(0+1+1, 0+0+1)/3 = (2/3, 1/3).$$

To get the overall centroid of R, you just weight these two centroids according to the ratio of the areas of the regions on which they are based to the total area of R. Since the rectangular part has 2/3'rds of the overall area,

$$(E(X), E(Y)) = (1/3)(2/3, 1/3) + (2/3)(1.5, .5) = (11/9, 4/9)$$

7 Conditional Distributions

Problem 7.1: Let X, Y have the joint PMF:

x \ y	1	2	3	4
1	.10	.05	.05	.05
2	.05	.10	.05	.05
3	.05	.05	.10	.05
4	.05	.05	.05	.10

(a) Compute $P(2 \le X \le 3|Y = 2)$ and $E(X|Y = 2)$.

Solution. Divide the second column by the column sum 0.25. The conditional PMF of X given $Y = 2$ is then

$$p_{X|Y}(x|y=2) = \begin{cases} 1/5, & x = 1 \\ 2/5, & x = 2 \\ 1/5, & x = 3 \\ 1/5, & x = 4 \end{cases}$$

Therefore:

$$P(2 \le X \le 3|Y = 2) = p_{X|Y}(2|2) + p_{X|Y}(3|2) = 3/5$$

$$\begin{aligned} E(X|Y=2) &= p_{X|Y}(1|2) * 1 + p_{X|Y}(2|2) * 2 + p_{X|Y}(3|2) * 3 + p_{X|Y}(4|2) * 4 \\ &= (1/5) * 1 + (2/5) * 2 + (1/5) * 3 + (1/5) * 4 = 12/5 \end{aligned}$$

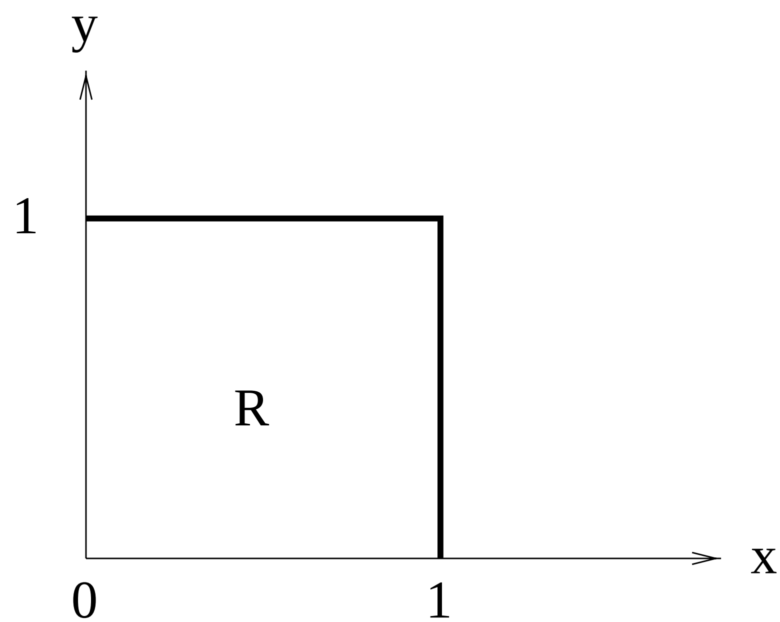
(b) Compute $P(Y \le 2|X = 4)$ and $E(Y|X = 4)$.

Solution. Divide the fourth row by the row sum 0.25. The conditional PMF of Y given $X = 4$ is then

$$p_{Y|X}(y|x=4) = \begin{cases} 1/5, & y = 1 \\ 1/5, & y = 2 \\ 1/5, & y = 3 \\ 2/5, & y = 4 \end{cases}$$

$$P(Y \leq 2|X = 4) = p_{Y|X}(1|4) + p_{Y|X}(2|4) = 2/5.$$

$$\begin{aligned} E(Y|X = 4) &= p_{Y|X}(1|4) * 1 + p_{Y|X}(2|4) * 2 + p_{Y|X}(3|4) * 3 + p_{Y|X}(4|x = 4) * 4 \\ &= (1/5) * 1 + (1/5) * 2 + (1/5) * 3 + (2/5) * 4 = 14/5 \end{aligned}$$



Problem 7.2: Let R be the region above. Let random variables X, Y have the joint density

$$f(x, y) = \begin{cases} x + y, & (x, y) \in R \\ 0, & \text{elsewhere} \end{cases}$$

(a) Compute $E(X|Y = y)$ as a function of y.

Solution. It's easy to show that $f_Y(y) = y + 1/2$ for $0 \leq y \leq 1$. Therefore, for each $0 \leq y \leq 1$, the conditional PDF $f(x|y)$ of X given $Y = y$ is given by:

$$f(x|y) = \begin{cases} \frac{x+y}{y+1/2}, & 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$E(X|Y = y) = \int_{-\infty}^{\infty} x f(x|y) dx = \int_0^1 x \frac{x + y}{y + 1/2} dx = \frac{1/3 + y/2}{y + 1/2}, \quad 0 \leq y \leq 1$$

(b) Compute $P(0.5 \leq Y \leq 1|X = 0.5)$.

Solution. Via symmetry, for each fixed x satisfying $0 \leq x \leq 1$, the conditional PDF $f(y|x)$ of Y given $X = x$ is given by

$$f(y|x) = \begin{cases} \frac{x+y}{x+1/2}, & 0 \leq y \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

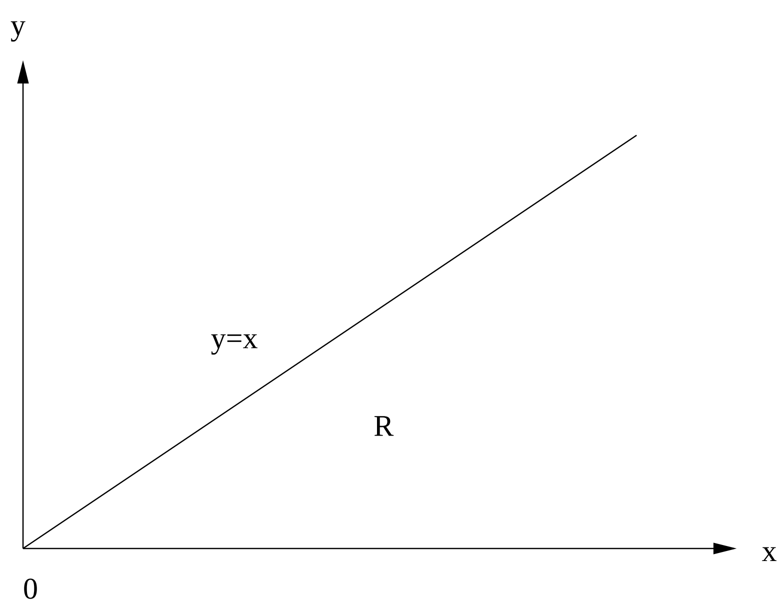
For $X = 0.5$, this ratio becomes

$$\frac{x+y}{x+1/2} = \frac{y+0.5}{0.5+0.5} = y + 0.5.$$

We have

$$\begin{aligned} P(0.5 \leq Y \leq 1|X = 0.5) &= \int_{0.5}^{1} f(y|x = 0.5)dy \\ &= \int_{0.5}^{1} (y + 0.5)dy \\ &= \left[y^2/2 + 0.5y\right]_{y=0.5}^{y=1} = 5/8 \end{aligned}$$

Problem 7.3: Let R be the infinite triangular region below.



Let $f(x, y)$ be the joint PDF of random variables X, Y as follows:

$$f(x,y) = \begin{cases} Ce^{-(x+y)}, & (x,y) \in R \\ 0, & \text{elsewhere} \end{cases}$$

(The value of the positive constant C is not needed in this problem.) Compute $E(Y|X = x)$ as a function of x.

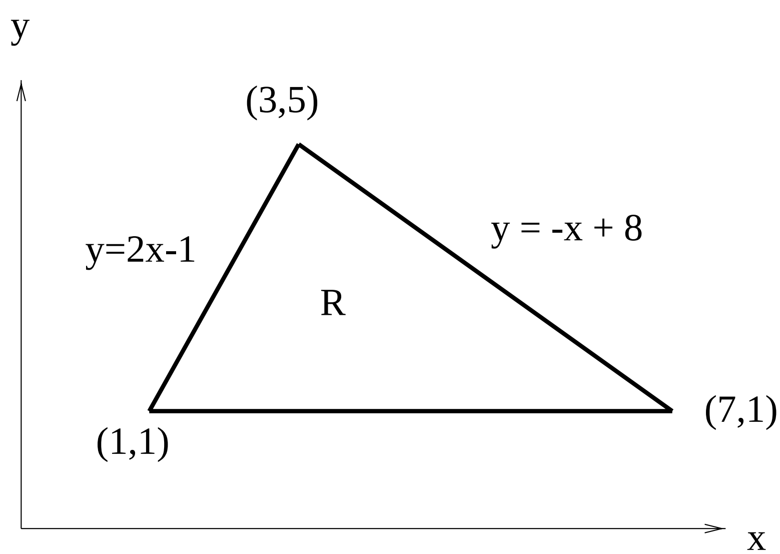
Solution. Taking a vertical slice at position x on the x-axis through R, we see that this slice goes from $y = 0$ to $y = x$. This determines the limits on our integrals when we compute $E(Y|X = x)$:

$$E(Y|X = x) = \frac{\int_0^x Cy \exp(-(x+y))dy}{\int_0^x C \exp(-(x+y))dy}$$

(Explanation: The denominator is simply $f_X(x)$; if we divide this into the joint density part of the integrand in the numerator, we see that this is the same as $\int y f(y|x)dy$.) Notice that C cancels and $\exp(-x)$ cancels. Therefore,

$$E(Y|X=x) = \frac{\int_0^x y\exp(-y)dy}{\int_0^x \exp(-y)dy} = \frac{1 - xe^{-x} - e^{-x}}{1 - e^{-x}}.$$

(The integral in the numerator was performed via integration by parts.)



Problem 7.4: We consider random variables X, Y jointly uniformly distributed in the triangular region R sketched above. The conditional mean function for X given Y takes the form

$$E[X|Y=y] = Ay + B, \;\; 1 \le y \le 5$$

Compute the constants A and B. (Hint: What is the geometric interpretation of this conditional mean function in terms of the region R?)

Solution. The line $x = Ay + B$ is the median extending from the vertex $(3,5)$ to the opposite side. It is halfway between the left edge $x = (y+1)/2$ and the right edge $x = -y + 8$. Therefore

$$Ay + B = (1/2)[(y+1)/2 - y + 8] = -y/4 + 17/4$$

Problem 7.5: Let X, Y be jointly uniformly distributed in the same semicircular region R used in Problem 2.2. Find the conditional mean functions $E[X|Y=y]$ and $E[Y|X=x]$ in an easy manner.

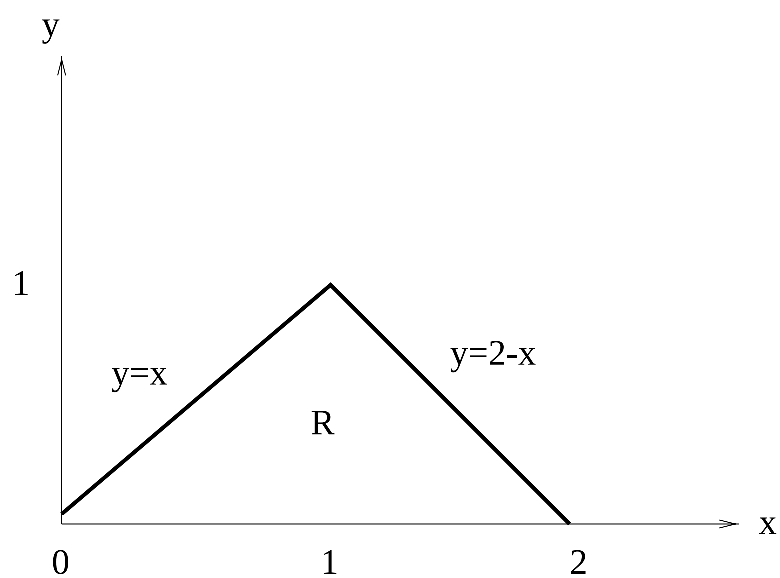
Solution. Since X, Y has a joint uniform distribution, the conditional distributions are all uniform. (For example, if $X = x$, then Y is conditionally uniformly distributed between 0 and $\sqrt{1-x^2}$.) The mean of a uniform distribution on an interval is the midpoint of the interval. You can visualize the mean function $E[Y|X = x]$ as a curve lying halfway between the upper boundary curve $y = \sqrt{1-x^2}$ of R and the lower boundary curve $y = 0$ of R. This gives us

$$E[Y|X = x] = (1/2)\sqrt{1-x^2}, \quad -1 \le x \le 1$$

Similarly, the mean function $E[X|Y = y]$ can be viewed as a curve midway between the left boundary curve $x = -\sqrt{1-y^2}$ of R and the right boundary curve $x = \sqrt{1-y^2}$ of R. This gives us

$$E[X|Y = y] = 0, \quad 0 \le y \le 1$$

Problem 7.6: Let R be the triangular region:



Let X, Y be jointly uniformly distributed over R. Find $E[X|Y = y]$ and $E[Y|X = x]$.

Solution. Let $f(x|y)$ be the conditional density of X given $Y = y$. Using the formula $f(x|y) = f_{X,Y}(x, y)/f_Y(y)$, one determines that for each y satisfying $0 < y < 1$:

$$f(x|y) = \begin{cases} 1/(2-2y), & y \le x \le 2-y \\ 0, & \text{elsewhere} \end{cases}$$

Taking a horizontal slice through the region R,

$$E[X|Y = y] = \int_y^{2-y} x f(x|y) dx = \int_y^{2-y} x/(2-2y) dx = 1, \quad 0 < y < 1$$

Because we have a uniform joint distribution in R, we can get $E[X|Y = y]$ by the following simpler method: Let $x = f_1(y)$ be the left boundary curve of R and let $x = f_2(y)$ be the right boundary curve of R (in this case, $f_1(y) = y$ and $f_2(y) = 2 - y$). If we take the curve halfway between these two boundary curves, namely,

$$x = (1/2)f_1(y) + (1/2)f_2(y),$$

it turns out that

$$E[X|Y = y] = (1/2)f_1(y) + (1/2)f_2(y) = (1/2)y + (1/2)(2 - y) = 1$$

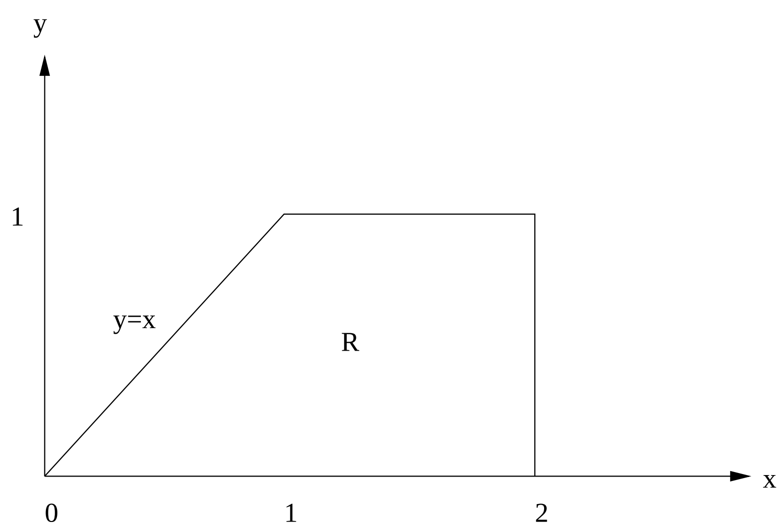
Similarly, if $y = g_1(x)$ is the upper boundary curve of R and $y = g_1(x)$ is the lower boundary curve of R, then

$$E[Y|X = x] = (1/2)g_1(x) + (1/2)g_2(x)$$

This gives us

$$E[Y|X = x] = \begin{cases} x/2, & 0 \le x \le 1 \\ (2 - x)/2, & 1 < x \le 2 \end{cases}$$

(Warning: Do not use this trick on any other joint distribution than the joint uniform distribution!)



Problem 7.7: (X, Y) is uniform over the preceding trapezoidal region R.

(a)

$$E(Y|X = x) =?, \quad 0 \le x \le 1.$$
$$E(Y|X = x) =?, \quad 1 \le x \le 2.$$

Solution. Fix x satisfying $0 \le x \le 1$. Then the vertical slice through R goes from $y = 0$ to $y = x$, and the conditional dist of Y given $X = x$ is uniform along this slice. The conditional mean $E(Y|X = x)$ is therefore the midpoint of the interval $[0, x]$, so that

$$E(Y|X = x) = x/2, \quad 0 \le x \le 1.$$

If we fix x so that $1 \le x \le 2$, the only difference is that the vertical slice goes from $y = 0$ to $y = 1$:

$$E(Y|X = x) = 1/2, \quad 1 \le x \le 2.$$

(b)

$$E(X|Y = y) =?, \quad 0 \le y \le 1.$$

Solution. Fix y satisfying $0 \le y \le 1$. Then, the horizontal slice through R goes from $x = y$ to $x = 2$:

$$E(X|Y = y) = (y + 2)/2, \quad 0 \le y \le 1.$$

(c)

$$f_X(x) =?, \quad 0 \le x \le 1.$$
$$f_X(x) =?, \quad 1 \le x \le 2.$$

Solution. The area of R is $3/2$. The joint density is the reciprocal of this, namely, $2/3$. Therefore, for $0 \le x \le 1$:

$$f_X(x) = \int_{y=0}^{y=x} (2/3)dy = 2x/3.$$

For $1 \le x \le 2$,

$$f_X(x) = \int_{y=0}^{y=1} (2/3)dy = 2/3.$$

(d)

$$f_Y(y) =?, \quad 0 \le y \le 1.$$

Solution.

$$f_Y(y) = \int_{x=y}^{x=2} (2/3)dx = (2/3)(2 - y).$$

8 Law of Iterated Expectation

Problem 8.1: A discrete random variable X has the PMF

$$p_X(x) = \begin{cases} 1/4, & x = 1 \\ 1/4, & x = 2 \\ 1/2, & x = 3 \end{cases}$$

A second random variable Y is unspecified, but it is known that

$$\begin{aligned} E[Y|X = x] &= (3 + 2x)/2 \\ \text{Var}[Y|X = x] &= 1/12 \end{aligned}$$

Try to compute $\rho_{X,Y}$ from the given information.

Solution. First, we can write

$$E[XY] = \sum_x E[XY|X = x]p_X(x)$$

Notice that

$$E[XY|X = x] = E[xY|X = x] = xE[Y|X = x] = x(3 + 2x)/2$$

Substituting this in the preceding, we obtain

$$E[XY] = \sum_{x=1}^{3} x[(3 + 2x)/2]p_X(x) = 9.125$$

If we now compute the means of X and Y, we will be able to compute the covariance between X and Y:

$$\begin{aligned} E[X] &= 1(1/4) + 2(1/4) + 3(1/2) = 9/4 \\ E[Y] &= \sum_x E[Y|X = x]p_X(x) = \sum_{x=1}^{3}[(3 + 2x)/2]p_X(x) = 3.75 \end{aligned}$$

This gives us

$$\mathrm{Cov}(X, Y) = E[XY] - \mu_X\mu_Y = 0.6875$$

If we now compute the variances of X and Y, we will be able to compute the correlation coefficient:

$$\begin{aligned} E[X^2] &= 1(1/4) + 4(1/4) + 9(1/2) = 23/4 \\ \mathrm{Var}[X] &= E[X^2] - \mu_X^2 = .6875 \\ E[Y^2] &= \sum_x E[Y^2|X = x]p_X(x) \\ &= \sum_x (\mathrm{Var}[Y|X = x] + \{E[Y|X = x]\}^2)p_X(x) \\ &= \sum_x [1/12 + (0.25)(3 + 2x)^2]p_X(x) = 14.8333 \\ \mathrm{Var}[Y] &= E[Y^2] - \mu_Y^2 = .7708 \end{aligned}$$

Finally,

$$\rho_{X,Y} = \sigma_{X,Y}/\sigma_X\sigma_Y = .6875/\sqrt{.6875}\sqrt{.7708} = .9444$$

It should be pointed out that the given conditional mean and variance for Y do not uniquely determine the distribution of (X, Y), and yet we were able to compute $\rho_{X,Y}$ from the given information. Here are two different random experiments that give rise to the given information in this problem:

Experiment 1: Select X according to the given PMF. Select, independently of X, a random Z which is uniformly distributed between 1 and 2. Take $Y = X + Z$.

Experiment 2: Select X according to the given PMF. Select, independently of X, a random Z which is Gaussian with mean 3/2 and variance 1/12. Take $Y = X + Z$.

Problem 8.2: Random variable X is Poisson with parameter $\lambda = 1$. A coin with probability of heads equal to e^{-X} is flipped 10 times, and Y is the number of heads.

(a) Find the regression function $E(Y|X = x)$.

Solution. If $X = x$, then the conditional distribution of Y is binomial(n, p), where $n = 10$ and $p = e^{-x}$. The conditional mean is therefore $n * p = 10e^{-x}$:

$$E(Y|X = x) = 10e^{-x}.$$

(b) Compute $E(Y)$.

Solution. By the law of iterated expectation,

$$E(Y) = \sum_x p_X(x)E(Y|X = x) = 10 \sum_x p_X(x)e^{-x} = 10E(e^{-X}).$$

For the Poisson distribution, we have

$$p_X(x) = e^{-\lambda}\lambda^x/x! = e^{-1}/x!, \quad x = 0, 1, 2, 3, \cdots.$$

Therefore,

$$\begin{aligned} E(Y) &= 10E(e^{-X}) \\ &= 10 \sum_{x=0}^{\infty} e^{-x}e^{-1}/x! \\ &= 10e^{-1} \sum_{x=0}^{\infty} (e^{-1})^x/x! \\ &= 10 * e^{-1} * \exp(e^{-1}) = 5.3146 \end{aligned}$$

Problem 8.3: Urn 1 contains 5 balls, two of which are numbered 1 and the remaining three of which are numbered 2. Urn 2 contains 7 balls, four of which are numbered 1 and the remaining three of which are numbered 2. An urn is chosen at random; random variable X is taken to be 1 if Urn 1 is chosen and is taken to be 2 if Urn 2 is chosen. (This constitutes Step 1 of a random experiment.) In the second step of the experiment, a ball is selected at random from the urn selected in Step 1, and RV Y is taken to be the number on the selected ball.

(a) Compute $E(Y|X = 1)$.

Solution.

$$E(Y|X = 1) = (2/5) * 1 + (3/5) * 2$$

(b) Compute $E(Y|X = 2)$.

Solution.

$$E(Y|X = 2) = (4/7) * 1 + (3/7) * 2$$

(c) Compute $E(Y)$. (Use law of iterated expectation.)

Solution.

$$E(Y) = (1/2)E(Y|X=1) + (1/2)E(Y|X=2)$$

Problem 8.4: Consider the following two-step experiment:

Step 1: Select a real number X at random from the interval $[0, 1]$.

Step 2: Flip a coin with $P(H) = X$ five times, and let Y be the number of heads obtained.

Compute the following:

(a) $E(Y)$

(b) $Var(Y)$

(c) $E(XY)$

Solution of (a). Given $X = x$, the conditional distribution of Y is binomial(n, p) where $n = 5$ and $p = x$. Therefore,

$$E(Y|X=x) = np = 5x.$$

Integrating,

$$E(Y) = \int_0^1 E(Y|X=x) f_X(x)dx = \int_0^1 5x f_X(x)dx = 5E(X) = 5/2.$$

Solution of (b). The conditional mean of Y given $X = x$ is np and the conditional variance is $np(1-p)$. (Check Chapter 2 for the mean and variance of the binomial(n, p) distribution.) The conditional second moment of Y is expressible as the sum of the conditional variance and the square of the conditional mean:

$$E(Y^2|X=x) = np(1-p) + (np)^2 = 5x(1-x) + (5x)^2 = 5x + 20x^2.$$

Integrating,

$$\begin{aligned} E(Y^2) &= \int_0^1 E(Y^2|X=x) f_X(x)dx \\ &= \int_0^1 (5x + 20x^2) f_X(x)dx \\ &= E(5X + 20X^2) = 5/2 + 20((1/12) + (1/2)^2) = 55/6 \end{aligned}$$

We conclude that

$$Var(Y) = E(Y^2) - \mu_Y^2 = (55/6) - (5/2)^2 = 35/12.$$

Solution of (c). Notice that

$$E(XY|X=x) = E(xY|X=x) = xE(Y|X=x) = x*np = x*5x = 5x^2.$$

Integrating,

$$\begin{aligned} E(XY) &= \int_0^1 E(XY|X=x) f_X(x)dx \\ &= \int_0^1 5x^2 f_X(x)dx \\ &= 5E(X^2) = 5((1/12)+(1/2)^2) = 5/3 \end{aligned}$$

9 Distribution of Function of Two or More RV's

Problem 9.1: Let X, Y be the discrete RV's proven to be independent in Problem 2.1. Find the PMF of $Z = X + Y$.

Solution: Using z-transforms,

$$\begin{aligned} Z[p_X] &= .40 + (.60)z^{-1} \\ Z[p_Y] &= .20 + (.30)z^{-1} + (.50)z^{-2} \\ Z[p_Z] &= Z[p_X]Z[p_Y] = .08 + (.24)z^{-1} + (.38)z^{-2} + (.30)z^{-3} \end{aligned}$$

Inverting, we conclude

$$p_Z(z) = \begin{cases} .08, & z=0 \\ .24, & z=1 \\ .38, & z=2 \\ .30, & z=3 \end{cases}$$

Problem 9.2: Independent discrete random variables X and Y have the PMF's:

$$p_X(x) = \begin{cases} 1/4, & x=1 \\ 1/4, & x=2 \\ 1/2, & x=3 \end{cases}$$

$$p_Y(y) = \begin{cases} 1/2, & y=-1 \\ 1/2, & y=1 \end{cases}$$

Find the PMF of the discrete random variable $W = 3X - 2Y$.

Solution. We can write $W = (3X) + (-2Y)$. This allows us to express the PMF of W as the convolution of the PMF's of the random variables $U = 3X$ and $V = -2Y$. The PMF's of U and V are readily determined to be:

$$p_U(u) = \begin{cases} 1/4, & u=3 \\ 1/4, & u=6 \\ 1/2, & u=9 \end{cases}$$

$$p_V(v) = \begin{cases} 1/2, & v = -2 \\ 1/2, & v = 2 \end{cases}$$

The z-transforms of these PMF's are:

$$\begin{aligned} Z[p_U] &= (1/4)z^{-3} + (1/4)z^{-6} + (1/2)z^{-9} \\ Z[p_V] &= (1/2)z^2 + (1/2)z^{-2} \end{aligned}$$

Multiplying these together, we get the z-transform of the PMF of W:

$$Z[p_W] = (1/8)z^{-1} + (1/8)z^{-4} + (1/4)z^{-7} + (1/8)z^{-5} + (1/8)z^{-8} + (1/4)z^{-11}$$

Inverting, we get

$$p_W(w) = \begin{cases} 1/8, & w = 1 \\ 1/8, & w = 4 \\ 1/8, & w = 5 \\ 1/4, & w = 7 \\ 1/8, & w = 8 \\ 1/4, & w = 11 \end{cases}$$

Problem 9.3: Find the PMF of the random variable X giving the number of heads on three tosses of a fair coin, using the convolution method (z-transform method).

Solution. Write $X = X_1 + X_2 + X_3$, where, for each $i = 1, 2, 3$,

$$X_i = \begin{cases} 1, & \text{toss } i \text{ is heads} \\ 0, & \text{toss } i \text{ is tails} \end{cases}$$

Then

$$\begin{aligned} Z[p_X(x)] &= \prod_{i=1}^{3} Z[p_{X_i}] \\ &= [0.5 + (0.5)z^{-1}]^3 \\ &= (1/8) + (3/8)z^{-1} + (3/8)z^{-2} + (1/8)z^{-3} \end{aligned}$$

Taking the inverse z-transform,

$$p_X(x) = (1/8) + (3/8)\delta[x-1] + (3/8)\delta[x-2] + (1/8)\delta[x-3]$$

which breaks down as

$$p_X(x) = \begin{cases} 1/8, & x = 0 \\ 3/8, & x = 1 \\ 3/8, & x = 2 \\ 1/8, & x = 3 \end{cases}$$

Since X has the binomial distribution with $n = 3$ and $p = 1/2$ (why?), we know that this is the right answer.

Problem 9.4: It is given that

- μ and λ are positive parameters.
- T is a random variable with the density $\lambda \exp(-\lambda t) u(t)$.
- X is a random variable independent of T with the density $\mu \exp(-\mu x) u(x)$.

(a) Determine the Laplace transform of the density of X.

(b) Determine the Laplace transform of the density of $-T$.

(c) Determine the Laplace transform of the density of $U \triangleq X - T$.

(d) Determine the density of U.

(e) Determine the density of $V \triangleq \max(0, U)$.

(f) Determine $E[V]$.

Solution.
(a)-(d)

$$\begin{aligned}
\text{transform PDF of } X &= \frac{\mu}{s+\mu} \\
\text{transform PDF of } -T &= \frac{\lambda}{\lambda - s} \\
\text{transform PDF of } U &= \frac{\mu\lambda}{(\mu+s)(\lambda-s)} \\
&= \frac{\mu\lambda}{\lambda+\mu}\left[\frac{1}{s+\mu} + \frac{1}{\lambda - s}\right] \\
\text{density of } U &= \frac{\mu\lambda}{\lambda+\mu}[\exp(-\mu t)u(t) + \exp(\lambda t)u(-t)]
\end{aligned}$$

(e)

$$F_V(v) = P[U \leq v]u(v)$$

Differentiating,

$$\begin{aligned}
f_V(v) &= P[U \leq 0]\delta(v) + f_U(v)u(v) \\
&= \frac{\mu\delta(v)}{\lambda+\mu} + \frac{\mu\lambda}{\lambda+\mu}\exp(-\mu v)u(v)
\end{aligned}$$

(f)

$$E[V] = \frac{\lambda}{\mu(\lambda+\mu)}$$

Problem 9.5: Let X, Y be independent, each uniformly distributed in the interval $[0, 1]$. Let $W = X + Y$. Use Laplace transforms to find the PDF of W.

Solution. We have

$$L[f_W] = L[f_X]L[f_Y] = L[f_X]^2,$$

where "L" denotes the Laplace transform operator. Since

$$f_X(x) = u(x) - u(x-1)$$

the Laplace transform is

$$L[f_X] = (1 - e^{-s})/s$$

Squaring

$$L[f_W] = (1/s^2) + e^{-2s}/s^2 - 2e^{-s}/s^2$$

Inverting,

$$f_W(w) = wu(w) + (w-2)u(w-2) - 2(w-1)u(w-1)$$

which simplifies to

$$f_W(w) = \begin{cases} w, & 0 \le w \le 1 \\ 2-w, & 1 < w \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

Problem 9.6: Let X, Y be independent random variables with exponential distributions:

$$\begin{aligned} f_X(x) &= \lambda_1 e^{-\lambda_1 x} u(x) \\ f_Y(y) &= \lambda_2 e^{-\lambda_2 y} u(y) \end{aligned}$$

where we suppose that $\lambda_2 > \lambda_1 > 0$. Let $W = X + Y$. Find the density of W.

Solution. Multiplying the Laplace transforms of the marginal densities of X and Y,

$$\begin{aligned} L[f_W] &= \left(\frac{\lambda_1}{s+\lambda_1}\right)\left(\frac{\lambda_2}{s+\lambda_2}\right) \\ &= \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left[\frac{1}{s+\lambda_1} - \frac{1}{s+\lambda_2}\right] \end{aligned}$$

Inverting,

$$f_W(w) = \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}(e^{-\lambda_1 w} - e^{\lambda_2 w})u(w)$$

Problem 9.7: Consider independent random variables X and Y, in which X is equidistributed over the set $\{0, 1, 2\}$ and Y is equidistributed over the set $\{0, 1\}$. Derive the PMF of the random variable $W = \max(X, Y)$.

Solution. Notice that

$$\begin{aligned}
(X,Y) = (0,0) &\Rightarrow W = 0 \\
(X,Y) = (0,1) &\Rightarrow W = 1 \\
(X,Y) = (1,0) &\Rightarrow W = 1 \\
(X,Y) = (1,1) &\Rightarrow W = 1 \\
(X,Y) = (2,0) &\Rightarrow W = 2 \\
(X,Y) = (2,1) &\Rightarrow W = 2
\end{aligned}$$

The probability $p(x, y)$ is equal to $(1/3)(1/2) = 1/6$ for each of the six (x, y) pairs given above. Therefore, it is evident that

$$\begin{aligned}
p_W(0) &= 1/6 \\
p_W(1) &= 3/6 \\
p_W(2) &= 2/6
\end{aligned}$$

Problem 9.8: Let W_1, W_2, W_3 be independent random variables having the same PMF, given by

$$p_{W_i}(w) = \begin{cases} 1/3, & w = 1 \\ 1/3, & w = 2 \\ 1/3, & w = 3 \end{cases}$$

Find the PMF of the random variable $W = \min(W_1, W_2, W_3)$.

Solution. W takes the values $1, 2, 3$. We have to compute the probabilities $p_W(1), p_W(2), p_W(3)$. We compute $p_W(3)$ first:

$$p_W(3) = P[W = 3] = P[W_1 = 3, W_2 = 3, W_3 = 3] = (P[W_i = 3])^3 = 1/27$$

Next, we compute $p_W(2)$. First, notice that

$$\begin{aligned}
P[W \in \{2,3\}] &= P[W_1 \in \{2,3\}, W_2 \in \{2,3\}, W_3 \in \{2,3\}] \\
&= (P[W_i \in \{2,3\}])^3 = (2/3)^3 = 8/27
\end{aligned}$$

Since we now know that

$$p_W(2) + p_W(3) = 8/27$$

we can conclude that

$$p_W(2) = 7/27$$

Finally,

$$p_W(1) = 1 - p_W(2) - p_W(3) = 19/27$$

Problem 9.9: Relay switch 1 has a lifetime (in hours) which is an exponentially distributed RV T_1 with parameter a. Relay switch 2 has a lifetime (in hours) which is an exponentially distributed RV T_2 with parameter b. The two switches are connected in series to form a relay circuit, and we suppose that the switches operate independently of each other. Suppose switch 1 has a mean lifetime of 1000 hours, and that switch 2 has a mean lifetime of 2000 hours. Compute the mean lifetime of the overall relay circuit.

Solution. Let T be the lifetime of the relay circuit. Then

$$T = \min(T_1, T_2).$$

Let us find the PDF of T. We can do this by differentiating the CDF $F_T(t)$ of T. Let t be a fixed number ≥ 0. We have

$$\begin{aligned}
F_T(t) &= P(T \leq t) \\
&= 1 - P(T > t) \\
&= 1 - P(\min(T_1, T_2) > t) \\
&= 1 - P(T_1 > t, \quad T_2 > t) \\
&= 1 - P(T_1 > t)P(T_2 > t) \\
&= 1 - e^{-at} * e^{-bt} \\
&= 1 - e^{-(a+b)t}
\end{aligned}$$

Differentiating, we conclude that the PDF $f_T(t)$ of T is:

$$f_T(t) = (a+b)e^{-(a+b)t}u(t).$$

Notice that this is an exponential distribution with parameter $a+b$. The mean lifetime $E[T]$ is therefore $1/(a+b)$:

$$a = 1/E(T_1) = 1/1000$$

$$b = 1/E(T_2) = 1/2000$$

$$E(T) = 1/(a+b) = 666.67 \text{ hours}.$$

Problem 9.10: Find the mean lifetime of the relay circuit when you have the same two switches in Problem 9.9 connected in parallel.

Solution. The lifetime of the resulting relay circuit is now

$$T = \max(T_1, T_2).$$

For $t \geq 0$, the CDF value $F_T(t)$ is computed as follows:

$$\begin{aligned}
F_T(t) &= P(T \leq t) \\
&= P(\max(T_1, T_2) \leq t) \\
&= P(T_1 \leq t, \quad T_2 \leq t) \\
&= P(T_1 \leq t)P(T_2 \leq t) \\
&= (1 - e^{-at})(1 - e^{-bt})
\end{aligned}$$

Differentiating via the product rule, the density of T is expressible for $t \geq 0$ as:

$$f_T(t) = be^{-bt}(1 - e^{-at}) + ae^{-at}(1 - e^{-bt}).$$

Simplifying, one obtains

$$f_T(t) = (ae^{-at} + be^{-bt} - (a+b)e^{-(a+b)t})u(t).$$

This is *not* an exponential density! But, we can still compute the mean lifetime of the circuit:

$$\begin{aligned} E(T) &= \int_0^\infty t f_T(t)dt \\ &= \int_0^\infty ate^{-at}dt + \int_0^\infty bte^{-bt}dt - \int_0^\infty (a+b)te^{-(a+b)t}dt \\ &= \frac{1}{a} + \frac{1}{b} - \frac{1}{a+b} \end{aligned}$$

If we again suppose that switch 1 has a mean lifetime of 1000 hours, and that switch 2 has a mean lifetime of 2000 hours, we easily compute the mean lifetime of the overall circuit to be 2333.33 hours.

Problem 9.11: Let X and Y be independent standard Gaussian RV's. Determine the PDF $f_R(r)$ of the RV

$$R = \sqrt{X^2 + Y^2}.$$

Solution. The density $f_R(r)$ will clearly be zero for $r < 0$, so we may assume that $r \geq 0$. We compute the value of the CDF $F_R(r)$:

$$\begin{aligned} F_R(r) &= P(X^2 + Y^2 \leq r^2) \\ &= \iint_S f_{X,Y}(x,y)dxdy \\ &= \iint_S (1/2\pi)e^{-(x^2+y^2)/2}dxdy, \end{aligned}$$

where S is the circular region lying inside the circle $x^2 + y^2 = r^2$ in the xy-plane. If we switch over to polar coordinates, then $x^2 + y^2$ becomes r^2 and $dxdy$ becomes $rdrd\theta$. (We are using r to represent the polar coordinate r and also the radius of our circle; this should not be confusing.)

$$\iint_S (1/2\pi)e^{-(x^2+y^2)/2}dxdy = (1/2\pi)\int_0^{2\pi}\int_0^r e^{-r^2/2}rdrd\theta = 1 - e^{-r^2/2}.$$

Differentiating, we see that

$$f_R(r) = re^{-r^2/2}u(r).$$

(This density function defines what is called a *Rayleigh distribution*.)

Problem 9.12: Let X_1 and X_2 be independent exponentially distributed RV's, each having the density

$$e^{-x}u(x).$$

Compute the PDF $f_X(x)$ of the RV $X = X_1 + X_2$.

Solution. We have

$$f_X(x) = (e^{-x}u(x)) * (e^{-x}u(x)).$$

The Laplace transform of the right hand side is

$$\left(\frac{1}{s+1}\right)\left(\frac{1}{s+1}\right) = \frac{1}{(s+1)^2}.$$

The inverse Laplace transform of this is therefore $f_X(x)$. Using a table of Laplace transforms, one determines that

$$f_X(x) = xe^{-x}u(x).$$

Problem 9.13: Let X_1 and X_2 be independent exponentially distributed RV's. This time we suppose that the densities are different:

$$\begin{aligned} f_{X_1}(x) &= e^{-x}u(x) \\ f_{X_2}(x) &= 2e^{-2x}u(x) \end{aligned}$$

Compute the PDF $f_X(x)$ of the RV $X = X_1 + X_2$.

Solution. We have

$$f_X(x) = (e^{-x}u(x)) * (2e^{-2x}u(x)).$$

The Laplace transform of the right hand side is

$$\left(\frac{1}{s+1}\right)\left(\frac{2}{s+2}\right) = \frac{2}{(s+1)(s+2)}.$$

Using partial fractions,

$$\frac{2}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{2}{s+2},$$

Taking the inverse Laplace transform, we obtain the density $f_X(x)$:

$$f_X(x) = 2(e^{-x} - e^{-2x})u(x).$$

Problem 9.14: Let X_1 be Gaussian with mean μ_1 and variance σ_1^2. Let X_2 be Gaussian with mean μ_2 and variance σ_2^2. Compute the PDF $f_X(x)$ of the RV $X = X_1 + X_2$, assuming that X_1, X_2 are independent.

Solution. The generating function representation of $f_{X_1}(x)$ is

$$e^{\mu_1 s + 0.5\sigma_1^2 s^2}.$$

The generating function representation of $f_{X_2}(x)$ is

$$e^{\mu_2 s + 0.5\sigma_2^2 s^2}.$$

Multiplying these, you get the generating function representation of $f_X(x)$:

$$e^{(\mu_1+\mu_2)s + 0.5(\sigma_1^2+\sigma_2^2)s^2}$$

This is the type of generating function you get in working with Gaussian RV's. Therefore, X must be Gaussian with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$.

10 Joint Gaussian Distribution

Problem 10.1: For a pair of joint Gaussian random variables X, Y, it is known that

$$\begin{aligned} \sigma_X &= 8 \\ E[Y|X=x] &= (-x/8) + 2 \\ E[X|Y=y] &= -2y \end{aligned}$$

Compute each of the following:

(a) μ_X

(b) μ_Y

(c) $\rho_{X,Y}$

(d) σ_Y

Solution: Taking the expected value of both sides of the second and third equation, you get

$$\begin{aligned} \mu_Y &= -\mu_X/8 + 2 \\ \mu_X &= -2\mu_Y \end{aligned}$$

Solving these simultaneously, you get

$$\begin{aligned} \mu_Y &= 8/3 \\ \mu_X &= -16/3 \end{aligned}$$

You also get the following two equations from looking at equations for conditional means on pages 193-194 of Yates-Goodman:

$$\begin{aligned} \rho\sigma_Y/\sigma_X &= -1/8 \\ \rho\sigma_X/\sigma_Y &= -2 \end{aligned}$$

Solving these simultaneously, you get

$$\begin{aligned} \rho &= -1/2 \\ \sigma_Y &= 2 \end{aligned}$$

Problem 10.2: Let X, Y be jointly Gaussian with the parameters

$$\begin{aligned} \mu_x &= 1 \\ \mu_y &= -2 \\ \sigma_x^2 &= 3 \\ \sigma_y^2 &= 2 \\ \rho_{x,y} &= -1/2 \end{aligned}$$

Determine the joint density of the two new random variables

$$\begin{aligned} U &= 2X - 3Y + 4 \\ V &= -X + 2Y - 3 \end{aligned}$$

Solution. All that one has to do is compute the parameters $\mu_u, \mu_v, \sigma_u^2, \sigma_v^2, \rho_{u,v}$, and then plug these in the general form of a joint Gaussian density given in the book. The determination of μ_u and μ_v is easy and shall not concern us here, as we just need to do the evaluations

$$\begin{aligned} \mu_u &= 2\mu_x - 3\mu_y + 4 \\ \mu_v &= -\mu_x + 2\mu_y - 3 \end{aligned}$$

We concentrate here on showing the reader a computationally efficient way in which to compute the remaining three parameters. First, write the equations defining U, V in terms of X, Y in matrix form:

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

Pick off the "coefficient matrix", which is the matrix

$$A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

It is known that the following equation holds:

$$\begin{bmatrix} \sigma_u^2 & \sigma_{u,v} \\ \sigma_{u,v} & \sigma_v^2 \end{bmatrix} = A \begin{bmatrix} \sigma_x^2 & \sigma_{x,y} \\ \sigma_{x,y} & \sigma_y^2 \end{bmatrix} A^T$$

This gives us the computation

$$\begin{bmatrix} \sigma_u^2 & \sigma_{u,v} \\ \sigma_{u,v} & \sigma_v^2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & -0.5\sqrt{6} \\ -0.5\sqrt{6} & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 44.6969 & -26.5732 \\ -26.5732 & 15.8990 \end{bmatrix}$$

We point out that the matrix computation method above is valid even when X, Y are not jointly Gaussian. In fact, the method holds for any pair of random variables X, Y.

Problem 10.3:

$$f_{X,Y}(x, y) = \frac{1}{2\pi} e^{-x^2/2} e^{-(y-2x)^2/2}.$$

(a) Find $f_X(x)$ and $f(y|x)$. (Hint: Factor!)
Solution. Joint Gaussian (X, Y) RV's with zero means have density which factors as

$$C \exp(-x^2/2\sigma_x^2) \exp(-(y - \mu_{x|y})^2/2\sigma_{x|y}^2),$$

for some positive constant C. The first exponential factor is $f_X(x)$ (up to a multiplicative constant), and the second exponential factor is the conditional density $f(y|x)$ (up to a multiplicative constant). Therefore, in our problem here,

$$f_X(x) = C_1 e^{-x^2/2}$$

$$f(y|x) = C_2 e^{-(y-2x)^2/2}$$

for constants C_1, C_2 (actually, both of these constants are equal to $1\sqrt{2\pi}$).

(b) $E(Y|X = x)$, $Var(Y|X = x)$ are?
Solution. By inspection,

$$E(Y|X = x) = \mu_{y|x} = 2x.$$

$$Var(Y|X = x) = \sigma_{x|y}^2 = 1.$$

(c) $E(XY) =?$ (Hint: Use $E(Y|X = x)$.)

$$E(XY|X = x) = xE(Y|X = x) = 2x^2.$$

Therefore, by the law of iterated expectation

$$E(XY) = \int_{-\infty}^{\infty} 2x^2 f_X(x)dx = 2E[X^2] = 2.$$

(By inspection, X is Gaussian$(0, 1)$ and therefore its second moment is 1.)

(d) $E(X|Y=0)$, $Var(X|Y=0)$ are?

Solution. Plugging $y=0$ into the joint density, we see that the conditional density $f(x|Y=0)$ takes the form

$$f(x|Y=0) = Ce^{-5x^2/2},$$

This is a Gaussian density with mean 0 and variance 1/5. Therefore,

$$E(X|Y=0) = 0, \quad Var(X|Y=0) = 1/5.$$

(e) $\rho_{X,Y} = ?$

Solution.

$$Var(X|Y=0) = 1/5 = \sigma_X^2(1-\rho^2).$$

Since $\sigma_X^2 = 1$, we conclude that $\rho = \pm\sqrt{4/5}$. Since the regression line $E(Y|X=x) = 2x$ determined earlier has positive slope, it is the $+$ sign we should take. Therefore,

$$\rho = \sqrt{4/5}.$$