Solved Problems on Random Processes/Mean Square Estimation

1 Mean Square Estimation

Problem 1.1: Let the input to a channel be RV X which is exponentially distributed with mean 1. Given $X = x$, let the conditional distribution of the output Y from the channel be exponential with mean $1/x$. The minimum mean square receiver generates an estimate $\hat{X}$ of X of form

$$\hat{X} = \frac{1}{AY + B},$$

for certain constants A, B. Evaluate A and B.

Solution. The joint density of (X, Y) is

$$f_X(x,y) = \begin{cases} xe^{-x(y+1)}, & x \geq 0, y \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

Therefore,

$$E(X|Y=y) = \frac{\int_0^\infty x^2 e^{-x(y+1)} dx}{\int_0^\infty x e^{-x(y+1)} dx} = \frac{2}{y+1}$$

Therefore,

$$A = B = 0.5.$$

Problem 1.2: Let the input to a channel be RV X with PDF

$$f_X(x) = xe^{-x^2/2} u(x).$$

Let the output from the channel be RV Y given by

$$Y = 4X + 2Z,$$

where Z is a RV independent of X whose PDF is

$$f_Z(z) = ze^{-z^2/2} u(z).$$

The minimum mean square straight line receiver generates an estimate $\hat{X}$ of X of form

$$\hat{X} = AY + B,$$

for certain constants A, B. Evaluate A and B.

Solution.

$$\mu_X = \mu_Z = \sqrt{\pi/2} = 1.2533$$
$$\sigma_X^2 = \sigma_Z^2 = 2 - \pi/2 = 0.4292$$
$$\mu_Y = 4\mu_X + 2\mu_Z = 6\sqrt{\pi/2} = 7.5199$$
$$\sigma_Y^2 = 16\sigma_X^2 + 4\sigma_Z^2 = 40 - 10\pi = 8.5841$$
$$Cov(X,Y) = 4\sigma_X^2 = 8 - 2\pi = 1.7168$$
$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = 0.8944$$
$$A = \rho\sigma_X/\sigma_Y = 0.2000$$
$$B = \mu_X - A\mu_Y = -0.2507$$

Problem 1.3: The input to a channel is a RV X with mean 1 and variance 1. There are two outputs Y_1, Y_2 from the channel:

$$\begin{aligned} Y_1 &= X + Z_1 \\ Y_2 &= X + Z_2 \end{aligned}$$

where Z_1, Z_2 constitute the "channel noise random variables" and satisfy:

- Z_1 and Z_2 each have mean 1 and variance 1.
- Z_1 and Z_2 are independent of each other.
- Z_1 and Z_2 are each independent of X.

(a) Find the constants a_1, a_2 so that the estimator

$$\hat{X} = a_1 Y_1 + a_2 Y_2$$

will make the mean square error $E[(X - \hat{X})^2]$ a minimum.

Solution. Using the orthogonality principle, you solve the equations

$$\begin{bmatrix} E[Y_1^2] & E[Y_1 Y_2] \\ E[Y_1 Y_2] & E[Y_2^2] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} E[XY_1] \\ E[XY_2] \end{bmatrix}.$$

$$E(Y_1) = E(X) + E(Z_1) = 2$$
$$Var(Y_1) = Var(X) + Var(Z_1) = 2$$
$$E(Y_1^2) = Var(Y_1) + \mu_{Y_1}^2 = 6$$

Similarly,

$$E(Y_2) = 2, \quad E(Y_2^2) = 6$$

We also have

$$Cov(X, Y_1) = Cov(X, X) + Cov(X, Z_1) = 1 + 0 = 1$$

$$E(XY_1) = Cov(X, Y_1) + \mu_X \mu_{Y_1} = 3$$

Similarly,

$$E(XY_2) = 3$$

Finally, we have

$$Cov(Y_1, Y_2) = Cov(X, X) + Cov(X, Z_1) + Cov(X, Z_2) + Cov(Z_1, Z_2) = 1+0+0+0 = 1$$

$$E(Y_1 Y_2) = Cov(Y_1, Y_2) + \mu_{Y_1} \mu_{Y_2} = 5$$

You now solve

$$\begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

The solutions are

$$a_1 = a_2 = 3/11.$$

(b) Find the constants b_1, b_2, b_3 so that the estimator

$$\hat{X} = b_1 Y_1 + b_2 Y_2 + b_3$$

will make $E[(X - \hat{X})^2]$ a minimum.

Solution. Using the orthogonality principle, you solve the equation

$$\begin{bmatrix} E[Y_1^2] & E[Y_1 Y_2] & E[Y_1] \\ E[Y_1 Y_2] & E[Y_2^2] & E[Y_2] \\ E[Y_1] & E[Y_2] & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} E[XY_1] \\ E[XY_2] \\ E[X] \end{bmatrix}$$

which becomes (using the parameters computed in (a)):

$$\begin{bmatrix} 6 & 5 & 2 \\ 5 & 6 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$$

The solutions are

$$\begin{aligned} b_1 &= 1/3 \\ b_2 &= 1/3 \\ b_3 &= -1/3 \end{aligned}$$

Problem 1.4: A WSS random process $X(t)$ has autocorrelation function $5/(\tau^2 + 9)$. Write some Matlab code to find the coefficient A such that $\hat{X}(3) = AX(1)$ is the optimum first-order MS predictor of $X(3)$ based on $X(1)$.

Solution.

```
tau=0:2;
autocorrelation=5./(tau.^2+9);
RX0=autocorrelation(1);
RX2=autocorrelation(3);
A=RX2/RX0
```

Problem 1.5: A WSS random process $X(t)$ has autocorrelation function $3.5-2\cos(\tau)$. Find the coefficient A such that $\hat{X}(6) = AX(3)$ is the optimum first-order MS predictor of $X(6)$ based on $X(3)$.

Solution. Compute $A = R_X(3)/R_X(0)$.

Problem 1.6: A WSS random process $X(t)$ has autocorrelation function $5/(\tau^2+9)$. Write some Matlab code to find the coefficients A, B such that $\hat{X}(4) = AX(1) + BX(2)$ is the optimum second-order MS predictor of $X(4)$ based on $X(1), X(2)$.

Solution.

```
tau=0:3;
autocorrelation=5./(tau.^2+9);
RX0=autocorrelation(1);
RX1=autocorrelation(2);
RX2=autocorrelation(3);
RX3=autocorrelation(4);
M=[RX0 RX1; RX1 RX0];
v=inv(M)*[RX3 RX2]';
A=v(1)
B=v(2)
```

Problem 1.7: A WSS random process $X(t)$ has autocorrelation function $3.5 - 2\cos(\tau)$. Find the coefficients A, B such that $\hat{X}(7) = AX(2) + BX(5)$ is the optimum second-order MS predictor of $X(7)$ based on $X(2), X(5)$.

Solution. The orthogonality principle gives you the following equations to solve:

$$\begin{bmatrix} R_X(0) & R_X(3) \\ R_X(3) & R_X(0) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} R_X(5) \\ R_X(2) \end{bmatrix}$$

Problem 1.8: A discrete-time process (X_n) is passed through an additive noise channel with channel noise process (Z_n). The processes X and Z are uncorrelated zero-mean WSS processes with the following autocorrelation functions:

$$R_X(\tau) = 10/2^{|\tau|}, \qquad R_Z(\tau) = 10\delta[\tau]$$

The channel output process is $Y_n = X_n + Z_n$.

(a) Find the constant A so that $E[(X_n - AY_n)^2]$ is a minimum.
Solution.

$$A = E[X_n Y_n]/E[Y_n^2] = R_X(0)/(R_X(0) + R_Z(0)) = 0.50$$

(b) Find the constant B so that $E[(X_n - BY_{n-1})^2]$ is a minimum.

Solution.

$$B = E[X_n Y_{n-1}]/E[Y_{n-1}^2] = R_X(1)/(R_X(0) + R_Z(0)) = 0.25$$

Problem 1.9: In the block diagram below,

$$X \to \boxed{\text{channel}} \to Y \to \boxed{\text{estimator}} \to \hat{X}$$

the input RV X and the channel satisfy

$$\begin{bmatrix} p_X(0) = 0.25 \\ p_X(1) = 0.40 \\ p_X(2) = 0.35 \end{bmatrix} \quad \begin{bmatrix} P[Y=0|X=0] = 0.5 & P[Y=1|X=0] = 0.5 & P[Y=2|X=0] = 0 \\ P[Y=0|X=1] = 0.5 & P[Y=1|X=1] = 0 & P[Y=2|X=1] = 0.5 \\ P[Y=0|X=2] = 0 & P[Y=1|X=2] = 0.5 & P[Y=2|X=2] = 0.5 \end{bmatrix}$$

The (nonlinear) least-squares estimator $\hat{X} = \hat{X}_{\text{LS}}$ minimizes the mean square estimation error $E[(X - \hat{X})^2]$ and takes the form

$$\hat{X}_{\text{LS}} = \begin{cases} \hat{X}_{\text{LS}}(0), & Y = 0 \\ \hat{X}_{\text{LS}}(1), & Y = 1 \\ \hat{X}_{\text{LS}}(2), & Y = 2 \end{cases}$$

Find $\hat{X}_{\text{LS}}(0), \hat{X}_{\text{LS}}(1), \hat{X}_{\text{LS}}(2)$.

Solution. The matrix of joint probabilities is

$$\begin{bmatrix} 1/8 & 1/8 & 0 \\ 0.20 & 0 & 0.20 \\ 0 & 0.175 & 0.175 \end{bmatrix}$$

Normalizing columns, the matrix of conditional probabilities for the X values given the Y values is then

$$\begin{bmatrix} 5/13 & 5/12 & 0 \\ 8/13 & 0 & 8/15 \\ 0 & 7/12 & 7/15 \end{bmatrix}$$

Using this conditional probability matrix, you get:

$$\begin{aligned} \hat{X}_{\text{LS}}(0) &= E[X|Y=0] = 8/13 \\ \hat{X}_{\text{LS}}(1) &= E[X|Y=1] = 7/6 \\ \hat{X}_{\text{LS}}(2) &= E[X|Y=2] = 22/15 \end{aligned}$$

Problem 1.10: In the block diagram below,

$$X[n] \to \boxed{\text{channel}} \to Y[n] = X[n] + Z[n] \to \boxed{\text{filter}} \to \hat{X}[n]$$

the signal $X[n]$ is WSS with autocorrelation function $2^{-|\tau|}$ and the channel noise process $Z[n]$ is WSS with $R_Z[\tau] = \delta[\tau]$; signal & channel noise are uncorrelated. You are going to design a two-tap *predictive Wiener filter*, whose output at time n is of the form

$$\hat{X}[n+1] = AY[n] + BY[n-1]$$

and minimizes the prediction error $E[(X[n+1] - \hat{X}[n+1])^2]$.

(a) The prediction error $X[n+1] - \hat{X}[n+1]$ must be uncorrelated with each of the observations $Y[n]$ and $Y[n-1]$. Using this fact, you can write down two linear equations

$$\begin{aligned} C_{1,1}A + C_{1,2}B &= C_{1,3} \quad (1) \\ C_{2,1}A + C_{2,2}B &= C_{2,3} \quad (2) \end{aligned}$$

involving the unknown filter tap weights A and B. Determine the six constants

$$C_{1,1}, C_{1,2}, C_{1,3}, C_{2,1}, C_{2,2}, C_{2,3}$$

Solution. From the equations

$$\begin{aligned} E[(X[n+1] - AY[n] - BY[n-1])Y[n]] &= 0 \\ E[(X[n+1] - AY[n] - BY[n-1])Y[n-1]] &= 0 \end{aligned}$$

you obtain

$$\begin{aligned} AR_Y[0] + BR_Y[1] &= R_X[1] \\ AR_Y[1] + BR_Y[0] &= R_X[2] \end{aligned}$$

from which one determines that

$$C_{1,1} = 2 \quad C_{1,2} = 1/2 \quad C_{1,3} = 1/2$$

$$C_{2,1} = 1/2 \quad C_{2,2} = 2 \quad C_{2,3} = 1/4$$

(b) Solve the equations (1)-(2) simultaneously for A and B.
Solution. You get $A = 7/30$ and $B = 1/15$.

2 Nonstationary Processes

Problem 2.1: Let $r(t)$ be the ramp function

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Let random variable U be uniformly distributed in the interval $[0, 1]$. Let

$$X(t), \quad -\infty < t < \infty$$

be the continuous-time random process in which

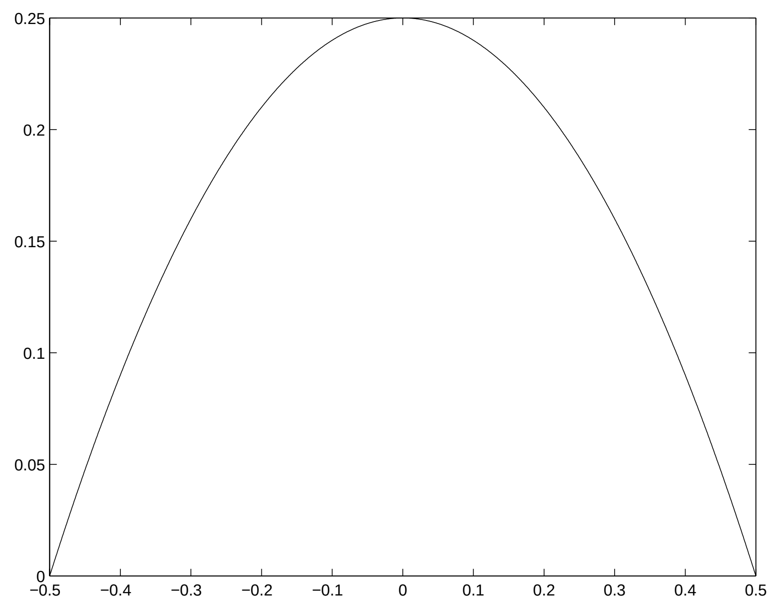
$$X(t) = r(U + t)r(U - t)$$

(a) Let the continuous-time signal $x(t)$ be the realization of the process $X(t)$ that you get when $U = 1/2$. Plot $x(t)$. Is $x(t) \geq 0$ for all t? Is $x(t)$ an even function of t? At what time t does $x(t)$ attain its peak value? At what two times t does $x(t)$ attain a value equal to one-half the peak value?

Solution. $x(t)$ can be described mathematically as:

$$x(t) = \begin{cases} 0.25 - t^2, & -0.5 \leq t \leq 0.5 \\ 0, & \text{elsewhere} \end{cases}$$

The plot is:



The signal is clearly nonnegative and even. The peak value (which is 0.25) is taken on when $t = 0$. Half the peak value (which is 0.125) is taken on at $t = \pm 1/\sqrt{8} = \pm 0.3536$.

(b) Compute $E(X(0))$ and compute $E(X(1/2))$. Can you conclude from these two answers that $X(t)$ is a nonstationary process?

Solution. For each t,

$$X(t) = \max(0, U^2 - t^2).$$

Thus,

$$\begin{aligned} X(0) &= U^2 \\ X(1/2) &= \max(0, U^2 - 1/4) \end{aligned}$$

$$E[X(0)] = E[U^2] = (1/12) + (1/2)^2 = 1/3$$

$$E[X(1/2)] = \int_0^1 \max(0, u^2 - 1/4)du = \int_{1/2}^1 (u^2 - 1/4)du = 1/6$$

The process must be nonstationary (if it were stationary, $E[X(t)]$ would be the same for all t).

Problem 2.2: A message M is modeled as an equiprobable discrete RV taking the values $1, 2, 4$. Frequency modulation is used to transmit M over a communication system. The resulting modulated FM wave is

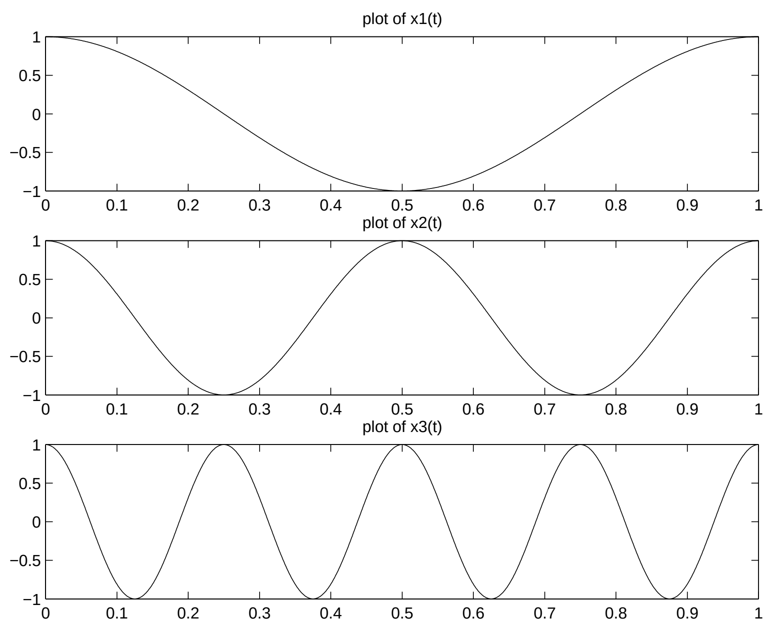
$$M(t) = \cos(2\pi t M), \quad -\infty < t < \infty.$$

(a) $M(t)$ is a continuous-time random signal. It has 3 realizations. What are each of these realizations? Each realization is a periodic signal; give the period of each realization. Plot each of the 3 realizations for $0 \leq t \leq 1$.

Solution. The 3 realizations are

$$\begin{aligned} x_1(t) &= \cos(2\pi t) \\ x_2(t) &= \cos(4\pi t) \\ x_3(t) &= \cos(8\pi t) \end{aligned}$$

The plots are:



The periods of $x_1(t), x_2(t), x_3(t)$ are $1, 1/2, 1/4$, respectively.

(b) Let X and Y be the RV's

$$X = M(1/4), \quad Y = M(1/8).$$

Compute $\rho_{X,Y}$.

Solution.

$$\begin{aligned} E(XY) &= E(\cos(\pi M/2)\cos(\pi M/4)) \\ &= (1/3)\cos(\pi/2)\cos(\pi/4) + (1/3)\cos(\pi)\cos(\pi/2) + (1/3)\cos(2\pi)\cos(\pi) \\ &= -1/3 \end{aligned}$$

$$\begin{aligned} E(X) &= E(\cos(\pi M/2)) = (1/3)(-1) + (1/3)(1) = 0 \\ E(X^2) &= (1/3)(-1)^2 + (1/3)(1)^2 = 2/3 \\ \sigma_X^2 &= 2/3 \end{aligned}$$

$$\begin{aligned} E(Y) &= E(\cos(\pi M/4)) = (1/3)(1/\sqrt{2}) + (1/3)(-1) = -0.0976 \\ E(Y^2) &= (1/3)(1/\sqrt{2})^2 + (1/3)(-1)^2 = 1/2 \\ \sigma_X^2 &= 1/2 - (-0.0976)^2 = 0.4905 \end{aligned}$$

$$\rho_{X,Y} = \frac{-1/3}{\sqrt{(2/3) * (0.4905)}} = -0.5829.$$

(c) Find a time t satisfying $0 < t < 1$ such that the RV's $M(t)$ and $M(1/4)$ are uncorrelated.

Solution. Since $E(M(1/4)) = 0$, the problem reduces to finding t for which

$$E(\cos(2\pi t M)\cos(\pi M/2)) = 0.$$

This is true if and only if

$$-\cos(4\pi t) + \cos(8\pi t) = 0.$$

$t = 1/2$ is obviously one solution between $t = 0$ and $t = 1$. Plotting the signal $-\cos(4\pi t) + \cos(8\pi t)$, you will see that this is the only solution.

(d) Is the process stationary?

Solution. $M(0)$ is identically equal to 1, i.e., it is not random. But, at other times, $M(t)$ can be random. Since all 1-D cross-sections do not have the same distribution, the process is not stationary.

Problem 2.3: Let A, B be independent RV's each having mean 0 and variance 1. Let $X(t)$ be the continuous-time random process

$$X(t) = At + (2t - 1)B, \quad -\infty < t < \infty$$

(a) Compute the autocorrelation function $R_X(s,t)$.

Solution.

$$R_X(s,t) = E[X(s)X(t)] = stE[A^2]+E[AB]\{s(2t-1)+t(2s-1)\}+E[B^2](2s-1)(2t-1).$$

Since $E[AB] = 0$ (why?), this simplifies to

$$R_X(s,t) = st + (2s-1)(2t-1).$$

(b) Find the value of t for which $\text{Var}[X(t)]$ is a minimum.

Solution. The mean function $\mu_X(t)$ is clearly zero. Therefore,

$$\text{Var}[X(t)] = R_X(t,t) = t^2 + (2t-1)^2.$$

Setting the first derivative equal to zero, one easily determines that $t = 2/5$.

(c) Find the value of t for which $E[X(t)X(t-1)]$ is a minimum.

Solution.

$$E[X(t)X(t-1)] = R_X(t,t-1) = t(t-1) + (2t-1)(2t-3).$$

Setting the derivative equal to zero yields $t = 9/10$.

(d) Is the process stationary?

Solution. No, because $R_X(s,t)$ is not a function of $s-t$.

Problem 2.4: Let A be a discrete RV with PMF

$$p_A(a) = \begin{cases} 1/2, & a = 0 \\ 1/4, & a = 1 \\ 1/4, & a = 2 \end{cases}$$

We consider the continuous-time random process $(X(t) : t \geq 0)$ in which

$$X(t) = \exp(-At), \quad t \geq 0.$$

(a) What are the realizations?

Solution. The three realizations are $u(t), \exp(-t)u(t), \exp(-2t)u(t)$.

(b) Find the PMF of the component RV $X(t)$.

Solution. For fixed $t > 0$, the RV $X(t)$ takes the three values 1, $\exp(-t)$, and $\exp(-2t)$, with probabilities 1/2, 1/4, 1/4, respectively. The RV $X(0)$ has to be treated as a special case. It takes just one value 1, with probability 1.

(c) Compute $E[X(t)]$.

Solution.

$$E[X(t)] = 1 * (1/2) + \exp(-t) * (1/4) + \exp(-2t) * (1/4)$$

(d) Does the limit of Var$[X(t)]$ exist as $t \to \infty$? If so, what is the limit?

Solution. As $t \to \infty$, the 1st realization stays at 1, and the 2nd and 3rd realizations converge to 0. In other words, we may regard $X(\infty)$ as the RV with the two values $1, 0$ taken on with probability 1/2 each. The mean of $X(\infty)$ is 1/2, its second moment is also 1/2 (since $X(\infty)^2 = X(\infty)$), and so its variance is $1/2 - (1/2)^2 = 1/4$.

We conclude that

$$\lim_{t\to\infty} \mathrm{Var}[X(t)] = \mathrm{Var}[X(\infty)] = 1/4$$

The long way to show this is first to show that

$$\begin{aligned}
\mathrm{Var}[X(t)] &= E[X(t)^2] - (0.5 + 0.25 * \exp(-t) + 0.25 * \exp(-2t))^2 \\
&= (0.5 + 0.25 * \exp(-2t) + 0.25 * \exp(-4t)) \\
&- (0.5 + 0.25 * \exp(-t) + 0.25 * \exp(-2t))^2
\end{aligned}$$

Taking the limit as $t \to \infty$, the exponentials drop out and you again get the answer 1/4.

3 WSS Processes

Problem 3.1: What approximate result will the following Matlab script give:

```
n=1:100000;
freq=pi/8;
theta=2*pi*rand(1,100000);
mean(cos(freq*n+theta).*cos(freq*(n+2)+theta))
```

Solution. Let X be the ergodic WSS discrete-time process for which

$$X_n = \cos(\Theta + n\pi/8),$$

where Θ is uniformly distributed between 0 and 2π. The Matlab script is using "time-averaging" to approximate $R_X(2)$. From "random sinusoid" theory, we know that

$$R_X(\tau) = (1/2)\cos(\omega_0\tau)$$

Therefore,

$$R_X(2) = (1/2)\cos(\pi/4) = 1/(2\sqrt{2}).$$

Problem 3.2: An ergodic WSS process $X(t)$ has autocorrelation function $5 + 4 * 2^{-|\tau|}$.

(a) Compute μ_X (assuming $\mu_X > 0$).

Solution. $\mu_X^2 = \lim_{\tau\to\infty} R_X(\tau) = 5$. Therefore, $\mu_X = \sqrt{5}$.

(b) Compute the power P_X generated by the process.

Solution. $P_X = R_X(0) = 9$.

(c) Compute $\text{Var}[X(t)]$.

Solution. $\text{Var}[X(t)] = P_X - \mu_X^2 = 4$.

(d) Compute $\text{Var}[X(1) + X(2) + X(3)]$.

Solution. From the two equations

$$\text{Cov}[X(i), X(j)] = 4 * 2^{-|i-j|}$$

and

$$\begin{aligned}\text{Var}[X(1) + X(2) + X(3)] &= \sum_{i=1}^{3} \text{Var}[X(i)] + 2\text{Cov}[X(1), X(2)] \\ &+ 2\text{Cov}[X(2), X(3)] + 2\text{Cov}[X(1), X(3)],\end{aligned}$$

we obtain:

$$\text{Var}[X(1) + X(2) + X(3)] = 3 * 4 + 2 * 2 + 2 * 2 + 2 * 1 = 22$$

(e) Compute the constant A so that $\hat{X}(2) = AX(1)$ will be the minimum mean square predictor of $X(2)$ based on $X(1)$.

Solution. We know from class that $A = R_X(1)/R_X(0)$. Therefore, $A = 7/9$.

Problem 3.3: A discrete-time ergodic WSS process X has a realization of period 3 which takes the consecutive left-to-right values of $1, 2, 3$ over one period. Compute μ_X, P_X, σ_X^2, $R_X(7)$ $R_X(11)$.

Solution. Since the realization is periodic with period 3, $R_X(\tau)$ will also be periodic with period 3. Using this fact and the ergodic property, we can compute all of the desired parameters via time averaging:

$$\mu_X = (1 + 2 + 3)/3 = 2.$$

$$P_X = R_X(0) = (1^2 + 2^2 + 3^2)/3 = 14/3.$$

$$\sigma_X^2 = P_X - \mu_X^2 = 2/3.$$

$$R_X(7) = R_X(7 - 2 * 3) = R_X(1) = (1/3)[1, 2, 3] \bullet [2, 3, 1] = 11/3.$$

$$R_X(11) = R_X(11 - 3 * 3) = R_X(2) = (1/3)[1, 2, 3] \bullet [3, 1, 2] = 11/3.$$

Problem 3.4: A box contains two coins, one fair and the other biased with probability of heads equal to 2/3. A coin is selected from the box at random and flipped forever. Let X be the discrete-time WSS process defined by:

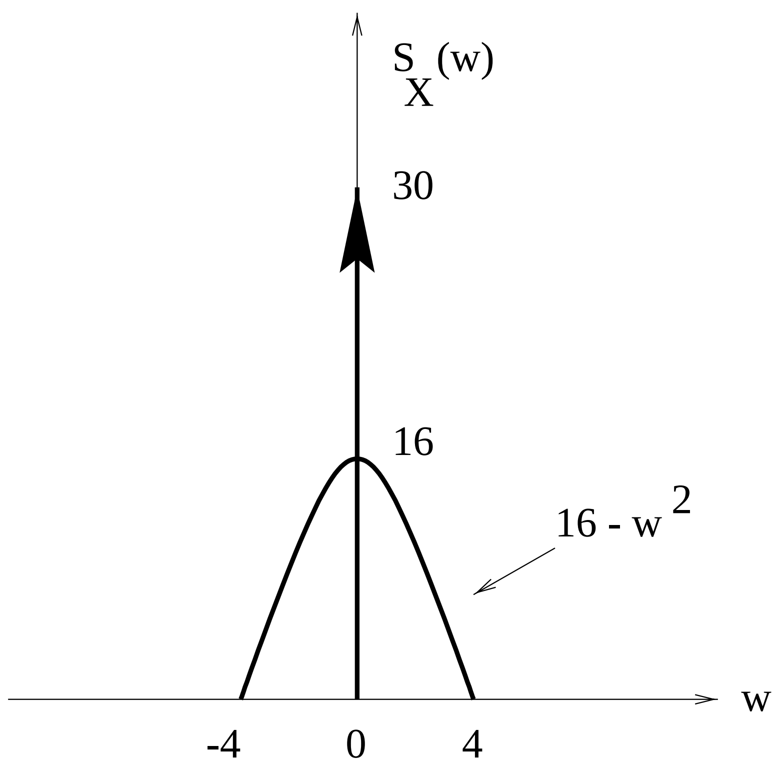
$$X[n] = \begin{cases} 1, & \text{if } n\text{−th flip is heads} \\ 0, & \text{if } n\text{−th flip is tails} \end{cases}$$

Work out $R_X(\tau), P_X, \mu_X, \sigma_X^2$. Is the process ergodic?

Solution.

$$
\begin{aligned}
P_X &= R_X(0) = (1/2)E[X[n]^2|\text{fair}] + (1/2)E[X[n]^2|\text{biased}] \\
&= (1/2)(1/2) + (1/2)(2/3) = 7/12 \\
R_X(\tau)\ (\tau \neq 0) &= (1/2)E[X[n]X[n+\tau]|\text{fair}] + (1/2)E[X[n]X[n+\tau]|\text{biased}] \\
&= (1/2)(1/2)(1/2) + (1/2)(2/3)(2/3) = 25/72 \\
\mu_X &= (1/2)E[X[n]|\text{fair}] + (1/2)E[X[n]|\text{biased}] \\
&= (1/2)(1/2) + (1/2)(2/3) = 7/12 \\
\sigma_X^2 &= P_X - \mu_X^2 = 7/12 - 49/144 = 35/144
\end{aligned}
$$

Intuitively, the process is not ergodic because half the time you get a realization according to the fair coin, and the other half of the time you get a realization according to the unfair coin. Time averages computed for these two types of realizations would give different answers.



Problem 3.5: The power spectral density $S_X(\omega)$ of an ergodic WSS process $X(t)$ is plotted above.

(a) Compute P_X.

Solution.

$$P_X = \frac{1}{2\pi}\int_{-4}^{4}(16-\omega^2)d\omega + \frac{30}{2\pi} = \frac{173}{3\pi}$$

(b) Determine the percentage of the total power that is due to the frequency band $-3 \leq \omega \leq 3$.

Solution.

$$\frac{1}{2\pi}\int_{-3}^{3}(16-\omega^2)d\omega + \frac{30}{2\pi} = \frac{54}{\pi}$$

$$\text{Percentage} = \left[\frac{54}{173/3}\right] \times 100 = 93.64\%$$

(c) Determine μ_X^2 and σ_X^2.

Solution. Only the inverse transform of the delta function part contributes anything as $\tau \to \infty$. So, μ_X^2 is the inverse transform of $30\delta(\omega)$, which is $15/\pi$.

$$\begin{aligned} \mu_X^2 &= 15/\pi \\ \sigma_X^2 &= P_X - 15/\pi = 128/3\pi \end{aligned}$$

Problem 3.6: A WSS process $X(t)$ has autocorrelation function $R_X(\tau) = 3 + 2\exp(-|\tau|)$.

(a) What is the average power generated by the process $X(t)$?

Solution. $P_X = R_X(0) = 5$

(b) Find $S_X(\omega)$.

Solution. Take Fourier transform of $R_X(\tau)$. You get

$$S_X(\omega) = 6\pi\delta(\omega) + \frac{4}{1+\omega^2}$$

(c) What percentage of the $X(t)$ power is due to frequencies in the band $-1 \leq \omega \leq 1$?

Solution.

$$(1/2\pi)\int_{-1}^{1} S_X(\omega)d\omega = 3 + 8\text{Tan}^{-1}(1)/2\pi = 4$$

Percentage of total power = 80%

Problem 3.7: Let $X(t)$ and $Y(t)$ be independent WSS processes with $R_X(\tau) = \cos(6\pi\tau)$ and $R_Y(\tau) = \cos(2\pi\tau)$. Compute the power spectral density of the process $Z(t) = X(t)Y(t)$. Compute the power generated by the process $Z(t)$ in two different ways.

Solution.

$$R_Z(\tau) = R_X(\tau)R_Y(\tau) = (1/2)[\cos(8\pi\tau) + \cos(4\pi\tau)]$$

$$S_Z(f) = (1/4)[\delta(f-4) + \delta(f+4) + \delta(f-2) + \delta(f+2)]$$

$$\begin{aligned} \text{power} &= R_Z(0) = 1 \\ \text{power} &= \int_{-\infty}^{\infty} S_Z(f)df = (1/4)[1+1+1+1] = 1 \end{aligned}$$

4 Linear Filtering of Processes

Problem 4.1: Let (X_n) be white noise with $R_X(\tau) = \delta[\tau]$ (the discrete-time delta function). Let (Y_n) be the WSS process arising from the causal filtering operation:

$$Y_n = 8X_n + (0.6)Y_{n-1}, \quad \text{for all integers } n \tag{3}$$

(a) Multiply both sides of (3) by X_n and take the expected value of both sides. Using the fact that $E[Y_i X_n] = 0$ for all $i < n$, deduce what the value of $E[X_n Y_n]$ is.

Solution.

$$E[Y_n X_n] = 8E[X_n^2] = 8$$

(b) Multiply both sides of (3) by Y_n and then by Y_{n-1}; take the expected value, and then solve the two resulting equations simultaneously for $R_Y(0)$ and $R_Y(1)$.

Solution. Multiplying as directed:

$$E[Y_n^2] = 8E[X_n Y_n] + (0.6)E[Y_{n-1}Y_n]$$

$$E[Y_n Y_{n-1}] = (0.6)E[Y_{n-1}^2]$$

These equations simplify to:

$$R_Y(0) = 64 + (0.6)R_Y(1)$$

$$R_Y(1) = (0.6)R_Y(0)$$

The solution is:

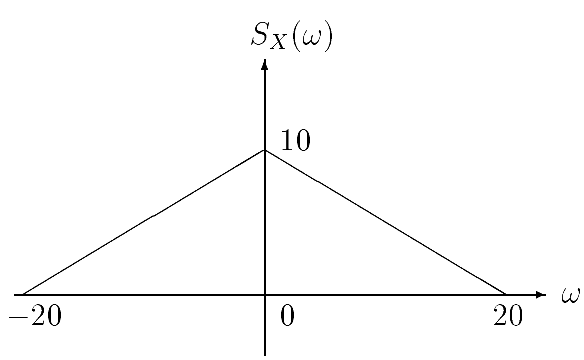
$$R_Y(0) = 100, \quad R_Y(1) = 60$$

(c) Multiply both sides of (3) by Y_{n-2}, take the expected value, and then compute $R_Y(2)$ from $R_Y(1)$.

Solution.

$$E[Y_n Y_{n-2}] = (0.6)E[Y_{n-1}Y_{n-2}]$$

$$R_Y(2) = (0.6)R_Y(1) = 36$$



Problem 4.2: The power spectral density $S_X(\omega)$ of a continuous-time WSS process $X(t)$ is plotted above.

(a) Compute the power P_X generated by the process $X(t)$.

Solution. Divide the area of the triangle by 2π.

$$P_X = (1/2) * 40 * 10 * (1/2\pi) = 100/\pi$$

(b) Let $Y(t)$ be the process obtained by linear filtering of $X(t)$ with filter frequency response function:

$$H(\omega) = \begin{cases} 1, & -10 \leq \omega \leq 10 \\ 0, & \text{elsewhere} \end{cases}$$

Compute the power P_Y generated by the process $Y(t)$.

Solution. You just have to remove two little triangles of base 10 and height 5 from the X process power spectrum.

$$P_Y = (200 - 2 * 0.5 * 10 * 5)/2\pi = 75/\pi$$

(c) Let $Y(t)$ be obtained from filtering of $X(t)$ as before, except that now the filter has frequency response

$$H(\omega) = \begin{cases} 1, & -B \leq \omega \leq B \\ 0, & \text{elsewhere,} \end{cases}$$

where the filter bandwidth B must be between 0 and 20. Compute the bandwidth B so that $P_Y/P_X = 0.90$.

Solution. You just have to remove two little triangles of base $20 - B$ and height $0.5(20 - B)$ from the X process power spectrum.

$$P_Y = (200 - 2 * 0.5 * 0.5 * (20 - B)(20 - B))/2\pi$$

Setting $P_Y = 0.9 * P_X$, you get

$$200 - (20 - B)^2/2 = 180$$

from which you determine that

$$B = 20 - \sqrt{40} = 13.68$$

Problem 4.3: A WSS process $X(t)$ has power spectral density function

$$S_X(\omega) = \begin{cases} 4 - (\omega^2/9), & |\omega| \leq 6 \\ 0, & \text{elsewhere} \end{cases}$$

(a) Find the average power generated by this process.

Solution.

$$P_X = \frac{1}{2\pi}\int_{-6}^{6}[4-\omega^2/9]d\omega = 16/\pi$$

(b) Find $R_X(\tau)$.

Solution.

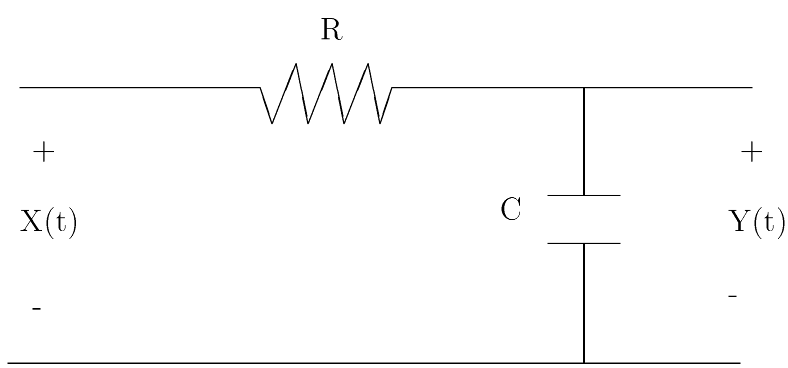
$$\begin{aligned} R_X(\tau) &= \frac{1}{2\pi}\int_{-6}^{6}[4-\omega^2/9]e^{j\omega\tau}d\omega \\ &= \frac{1}{\pi}\int_{0}^{6}[4-\omega^2/9]\cos(\tau\omega)d\omega \\ &= \frac{4\sin(6\tau)}{\pi\tau} + (d^2/d\tau^2)\left[\frac{\sin(6\tau)}{9\pi\tau}\right] \end{aligned}$$

(c) $X(t)$ is applied as input to the RC filter given in the diagram below (take $RC = 1/6$). Find the average power generated by the filter output process $Y(t)$.

Solution.

$$S_Y(\omega) = |H(\omega)|^2 S_X(\omega) = \left[\frac{1}{1+\omega^2/36}\right]S_X(\omega)$$

$$\begin{aligned} P_Y &= \frac{1}{2\pi}\int_{-6}^{6}\frac{4-\omega^2/9}{1+\omega^2/36}d\omega \\ &= \frac{4}{\pi}\int_{0}^{6}\left[\frac{72}{36+\omega^2}-1\right]d\omega \\ &= 12\left(1-\frac{2}{\pi}\right) \end{aligned}$$



Problem 4.4: Refer to the RC filter diagram above. Suppose that $RC = 2$.

(a) Assuming $\mu_X = 2$. Compute μ_Y.

Solution.

$$H(s) = \frac{1/2}{s+1/2}$$

$$h(t) = 0.5\exp(-t/2)u(t)$$
$$\mu_Y = \mu_X \int_{-\infty}^{\infty} h(t)dt = 5 * 1 = 5$$

(b) Assuming $R_X(\tau) = 6\delta(\tau)$, compute P_Y.
Solution. Since the input is white,

$$P_Y = P_X \int_{-\infty}^{\infty} h(t)^2 dt = 6 \int_0^{\infty} 0.25\exp(-t)dt = 1.5.$$

Problem 4.5: A WSS process $X(t)$ has autocorrelation function $R_X(\tau) = A + B\delta(\tau)$, where A and B are constants. Let $Y(t)$ be the process defined by

$$Y(t) = \int_{t-3}^{t} X(\alpha)d\alpha$$

for all t.

(a) Find $R_Y(\tau)$.
Solution.

$$\begin{aligned} h(\tau) &= u(\tau) - u(\tau - 3) \\ h(\tau) * h(-\tau) &= \phi(\tau) \end{aligned}$$

where

$$\phi(\tau) = \begin{cases} 3\left[1 - \frac{|\tau|}{3}\right], & -3 \le \tau \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

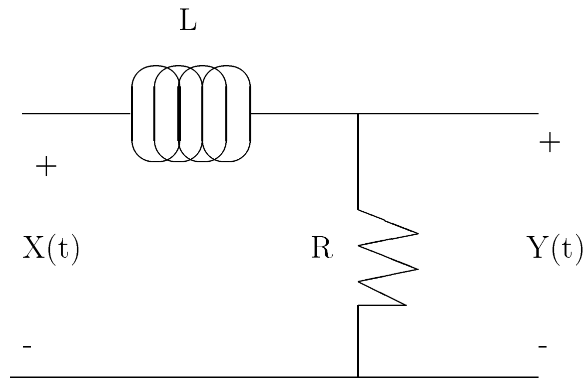
$$\begin{aligned} R_Y(\tau) &= \phi(\tau) * R_X(\tau) \\ &= A\int \phi(\tau)d\tau + B\phi(\tau) \\ &= 9A + B\phi(\tau) \end{aligned}$$

(b) Find $S_Y(\omega)$.
Solution.

$$S_Y(\omega) = \mathcal{F}[R_Y(\tau)] = 18\pi A\delta(\omega) + B\left[\frac{2\sin(3\omega/2)}{\omega}\right]^2$$

(c) Find the average power generated by the process $Y(t)$.
Solution.
$$P_Y = R_Y(0) = 9A + B\phi(0) = 9A + 3B$$

(d) Find the mean μ_Y of the process $Y(t)$.
Solution.
$$\mu_Y = \mu_X\left[\int h(t)dt\right] = \pm 3\sqrt{A}$$



Problem 4.6: The input $X(t)$ to the filter above is WSS with autocorrelation function

$$R_X(\tau) = 6\delta(\tau) + 10, \quad -\infty < \tau < \infty$$

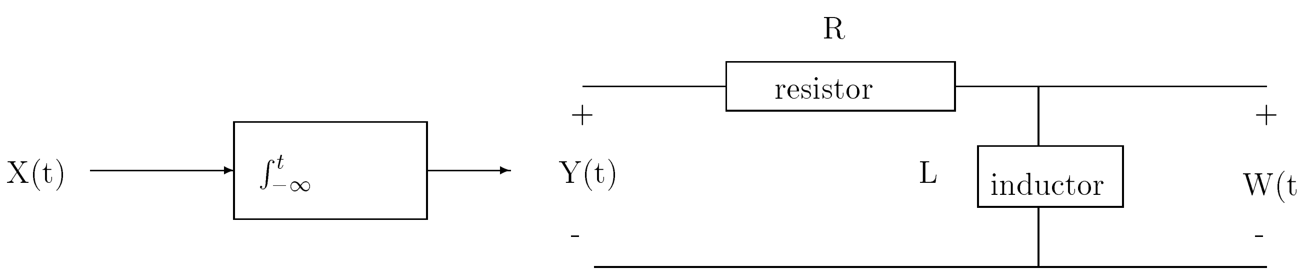
(a) Find μ_X and μ_Y.
Solution.

$$\begin{aligned} h(t) &= (R/L)e^{-tR/L}u(t) \\ \mu_X &= \pm\sqrt{10} \\ \mu_Y &= \mu_X \int_{-\infty}^{\infty} h(t)dt = \pm\sqrt{10} \end{aligned}$$

(b) Find output power.
Solution.

$$\begin{aligned} P_Y &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 S_X(\omega)d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{R^2}{L^2\omega^2 + R^2}\right)(6 + 20\pi\delta(\omega))d\omega \\ &= 3R/L + 10 \end{aligned}$$

Problem 4.7: In the block diagram below



$X(t)$ is white noise with $R_X(\tau) = A\delta(\tau)$.

(a) Find $S_W(\omega)$.

Solution. Let $H(\omega)$ be the frequency response of the circuit part of the overall system.

$$\begin{aligned} S_Y(\omega) &= \frac{A}{\omega^2} \\ H(\omega) &= \frac{Lj\omega}{Lj\omega + R} \\ |H(\omega)|^2 &= \frac{L^2\omega^2}{L^2\omega^2 + R^2} \\ S_W(\omega) &= |H(\omega)|^2 S_Y(\omega) = \frac{AL^2}{L^2\omega^2 + R^2} \end{aligned}$$

(b) Find $R_W(\tau)$.

Solution. Take the inverse Fourier transform of $S_W(\omega)$:

$$R_W(\tau) = \frac{AL}{2R}\exp(-R|\tau|/L)$$

(c) Compute P_W two different ways.

Solution.

$$\begin{aligned} P_W &= R_W(0) = \frac{AL}{2R} \\ P_W &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_W(\omega)d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{A}{\omega^2 + (R/L)^2} d\omega = AL/2R \end{aligned}$$

Problem 4.8: A band-limited continuous-time WSS process $x(t)$ has power spectral density

$$S_x(f) = \begin{cases} |f|, & -B \le f \le B \\ 0, & \text{elsewhere} \end{cases}$$

Compute the bandwidth B so that if $x(t)$ is passed through a differentiator, the power at the differentiator output will be equal to the power at the differentiator input.

Solution.

$$\begin{aligned} \text{input power} &= 2\int_0^B S_x(f)df = 2\int_0^B f\,df \\ \text{output power} &= 2\int_0^B (2\pi f)^2 S_x(f)df = 8\pi^2\int_0^B f^3 df \end{aligned}$$

Solving for B you get

$$B = \frac{1}{\sqrt{2}\pi}$$

Problem 4.9: A zero mean WSS process $x[n]$ has autocorrelation function

$$R_x(\tau) = \begin{cases} 1, & \tau = 0 \\ 1/2, & \tau = \pm 1 \\ 0, & \text{elsewhere} \end{cases}$$

Let $y[n]$ be the process

$$y[n] = \alpha[x[n] + x[n-1]]$$

where α is a parameter that will be determined.

(a) Find the impulse response of the system that carries $x[n]$ into $y[n]$ and use this to find the autocorrelation function of $y[n]$.

Solution. $h[n] = \alpha, \;\; n = 0, 1$ (zero elsewhere) and so

$$R_y[\tau] = h[\tau] * h[-\tau] * R_x(\tau)$$

The right side of the above equation in z domain is

$$\alpha^2(1 + z^{-1})(1 + z)(1 + .5z^{-1} + .5z) = \alpha^2(.5z^{-2} + 2z^{-1} + 3 + 2z + .5z^2)$$

Therefore

$$R_y(\tau) = \begin{cases} 3\alpha^2, & \tau = 0 \\ 2\alpha^2, & \tau = \pm 1 \\ (.5)\alpha^2, & \tau = \pm 2 \\ 0, & \text{elsewhere} \end{cases}$$

(b) Find the value of α that minimizes $E[\{x[n] - y[n-1]\}^2]$.

Solution. It is desired to compute the power generated by the process $z[n] = x[n] - y[n-1]$. You get the process $z[n]$ by putting $x[n]$ thru a filter with transfer function $1 - \alpha z^{-1} - \alpha z^{-2}$. So, the z-transform of $R_z(\tau)$ is

$$(1 - \alpha z^{-1} - \alpha z^{-2})(1 - \alpha z - \alpha z^2)(1 + .5z^{-1} + .5z) = (3\alpha^2 - \alpha + 1) + \text{other terms}$$

We conclude

$$E[(x[n] - y[n-1])^2] = R_z(0) = 3\alpha^2 - \alpha + 1$$

This is minimized when $\alpha = 1/6$.

(c) Compute the average power generated by the process $y[n]$, using the value of α you found in (b).

Solution. (c) $3\alpha^2 = 1/12$.

Problem 4.10: Let $x(t)$ be a zero-mean WSS process with

$$S_x(f) = \exp(-\pi|f|)$$

Compute

$$E\left[\left\{\int_{t-1}^{t} x(u)du\right\}^2\right]$$

Solution. You must find $R_y(0)$ where $y(t)$ is the process

$$y(t) = h(t) * x(t)$$

and where

$$h(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & \text{elsewhere} \end{cases}$$

Letting $Q(t) = h(t) * h(-t)$, we have

$$\begin{aligned} R_y(0) &= \int_{-\infty}^{\infty} Q(\tau) R_x(\tau) d\tau \\ &= 2\int_0^1 (1 - |\tau|)\frac{2}{\pi(1+4\tau^2)} d\tau = \frac{4}{\pi}\left[\frac{\text{Tan}^{-1}2}{2} - \frac{\ln 5}{8}\right] \end{aligned}$$

Problem 4.11: Let $X(t)$, $-\infty < t < \infty$, be a white noise process and let $Y(t)$ be the process

$$Y(t) = X(t+1) + X(t-1)$$

(a) Find $S_Y(f)$.

Solution. $H(f) = 2\cos(2\pi f)$ and so

$$S_Y(f) = |H(f)|^2 S_X(f) = 4\sigma^2 \cos^2(2\pi f)$$

(b) Pass $Y(t)$ through an ideal low-pass filter of bandwidth 1 Hz. Compute the output power.

Solution. output power $= \int_{-1}^{1} 4\sigma^2 \cos^2(2\pi f) df = 4\sigma^2$

(c) Pass $Y(t)$ through an ideal band-pass filter whose passband is $\{f : |f-4| \le 1/3\}$. Compute the output power.

Solution. output power $= 2\int_{4-1/3}^{4+1/3} 4\sigma^2 \cos^2(2\pi f) df = 8\sigma^2/3$

Problem 4.12: A white noise process with autocorrelation function equal to the delta function is the input to a linear time-invariant system. Compute the power generated by the output process if:

(a) The system is discrete-time with impulse response $h[n] = 3^{-n}u[n]$.

Solution. $R_y(0) = \sum_n h[n]^2 = \sum_{n=0}^{\infty} 9^{-n} = 9/8$.

(b) The system is continuous-time with impulse reponse $h(t) = \exp(-3t)u(t)$.

Solution. $R_y(0) = \int h(t)^2 dt = \int_0^{\infty} \exp(-6t) dt = 1/6$.

Problem 4.13: An ergodic WSS process $X(t)$ satisfies

$$R_X(\tau) = 1 + \cos\tau.$$

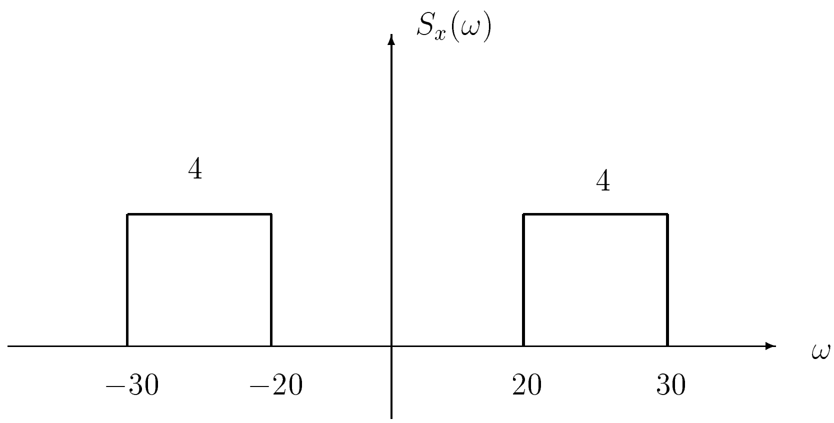
Let $Z(t)$ be the process

$$Z(t) = \int_{t-1}^{t} (\sin u)X(u)du.$$

Assuming that $\mu_X < 0$, find the mean function $\mu_Z(t)$.

Solution. Time-averaging $R_X(\tau)$, we see that $\mu_X^2 = 1$. Since $\mu_X < 0$, $\mu_X = -1$.

$$\mu_Z(t) = E[Z(t)] = \int_{t-1}^{t} (\sin u)(\mu_X)du = \cos t - \cos(t-1).$$



Problem 4.14: A WSS process $X(t)$ has power spectral density plotted above. Let $Y(t)$ be the WSS process

$$Y(t) = \int_{-\infty}^{t} X(u)du, \quad -\infty < t < \infty$$

(a) Compute the power generated by the process $X(t)$.

Solution. The total area of the rectangles is 80. Dividing by 2π,

$$P_X = \frac{40}{\pi}.$$

(b) Compute the power generated by the process $Y(t)$.

Solution.

$$P_Y = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(\omega)d\omega = \frac{1}{\pi}\int_{20}^{30} 4|H(\omega)|^2 d\omega = \frac{1}{\pi}\int_{20}^{30} \frac{4}{\omega^2}d\omega = \frac{1}{15\pi}.$$

Problem 4.15: Let $X(t)$ be WSS with $R_X(\tau) = \exp(-2|\tau|)$. Let $Y(t)$ be the result of passing $X(t)$ through an LTI filter with impulse response $h(t) = \exp(-t)u(t)$. Compute $E[(X(t) - Y(t))^2]$.

Solution. Let $E(t)$ be the process $E(t) = X(t)-Y(t)$. Then $E(t) = (\delta(t)-h(t))*X(t)$, and so

$$S_E(\omega) = |1 - H(\omega)|^2 S_X(\omega)$$

Plugging in

$$H(\omega) = \frac{1}{j\omega + 1}$$
$$S_X(\omega) = \frac{4}{\omega^2 + 4}$$

you get

$$S_E(\omega) = \frac{4\omega^2}{(\omega^2+1)(\omega^2+4)} = \frac{-4/3}{\omega^2+1} + \frac{16/3}{\omega^2+4}$$
$$P_E = (1/2\pi)(-4/3)[\int_{-\infty}^{\infty} \frac{1}{\omega^2+1} d\omega - 4 \int_{-\infty}^{\infty} \frac{1}{\omega^2+4} d\omega] = 2/3$$

Problem 4.16: A WSS process (X_n) has power spectral density function

$$S_X(\omega) = \frac{1}{5 - 4\cos\omega}$$

Let (Y_n) be the WSS process in which

$$Y_n = X_n + X_{n-1}$$

for every n. Find $S_Y(\omega)$ and P_Y.

Solution. The transfer function is $H(z) = 1 + z^{-1}$ and so the frequency response function is $H(\omega) = 1 + \exp(-j\omega)$. This gives

$$|H(\omega)|^2 = 2(1 + \cos\omega)$$

$$\begin{aligned} S_Y(\omega) &= S_X(\omega)|H(\omega)|^2 \\ &= \frac{2(1+\cos\omega)}{5 - 4\cos\omega} \\ &= (-1/2) + \frac{9/2}{5 - 4\cos\omega} \end{aligned}$$

$$P_Y = (1/2\pi)\int_{-\pi}^{\pi} S_Y(\omega) d\omega = -1/2 + (9/2)R_X(0)$$

Using tables,

$$R_X(\tau) = \mathcal{F}^{-1}\left[\frac{1}{5 - 4\cos\omega}\right] = (1/3)(0.5)^{-|\tau|}$$

So,

$$P_Y = -1/2 + (9/2)(1/3) = 1$$

Problem 4.17: In an additive noise channel, the signal is

$$X(t) = 3 \sin 2t$$

and the noise $Z(t)$ has power spectrum

$$S_Z(\omega) = \begin{cases} 10, & -1 \leq w \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

Send the process $X(t) + Z(t)$ process through a differentiator.

(a) Compute the SNR in decibels at the differentiator input.

Solution. The power generated by a deterministic sinusoid is 1/2 the square of the amplitude. Therefore,

$$P_X = 3^2/2 = 4.5.$$

We get the noise power by integrating the noise power spectrum and dividing by 2π. Therefore,

$$P_Y = 10 * 2/2\pi = 10/\pi.$$

The differentiator input SNR is therefore

$$10 \log_1 0 \frac{4.5}{10/\pi} = 1.5 \text{ decibels}$$

(b) Compute the SNR in decibels at the differentiator output.

Solution. At the differentiator output, the signal part is $6 \cos 2t$ and its power is $36/2 = 18$. At the differentiator output, the noise part has power

$$(1/2\pi) \int_{-\infty}^{\infty} \omega^2 S_Z(\omega) d\omega = (1/2\pi) \int_{-1}^{1} 10\omega^2 d\omega = 10/3\pi$$

The SNR is therefore

$$10 \log_{10} \frac{18}{10/3\pi} = 12.3 \text{ decibels}$$

Problem 4.18: A discrete-time WSS process X has power spectral density

$$S_X(\omega) = 4\pi\delta(\omega - \pi/3) + 3\pi\delta(\omega - 3\pi/4), \quad 0 \leq \omega \leq \pi$$

Let Y be the discrete-time WSS process obtained by passing X through the discrete-time LTI filter with frequency response function

$$H(\omega) = \frac{5}{2 + \exp(-j\omega)}$$

(a) Compute the power P_X generated by process X.

Solution. Integrating $S_X(\omega)$ from 0 to π and dividing by 2π gives half the power (because the integral from $-\pi$ to 0 gives the other half). So $P_Y = 7$.

(b) Compute the power P_Y generated by process Y.

Solution.

$$|H(\omega)|^2 = \frac{25}{5 + 4\cos\omega}$$

$$|H(\pi/3)|^2 = 25/7$$

$$|H(3\pi/4)|^2 = 25/(5 - 2\sqrt{2}) = 11.51$$

$$P_Y = (1/\pi)(4\pi)(25/7) + (1/\pi)(3\pi)(11.51) = 48.8$$

Problem 4.19: A continuous-time ergodic WSS process $X(t)$ has aucorrelation function

$$R_X(\tau) = \frac{72\tau^2 + 36}{12\tau^2 + 1}.$$

It is processed as follows:

$$X(t) \to \boxed{\int_{t-8}^{t}} \to Y(t)$$

(a) Find power P_X generated by process X.

Solution. $P_X = R_X(0) = 36$.

(b) Find μ_X^2.

Solution.

$$\mu_X^2 = \lim_{\tau\to\infty} R_X(\tau) = 72/12 = 6$$

(c) Find μ_Y^2.

Solution. The impulse response function $h(t)$ is a rectangular pulse of amplitude 1 starting at time $t = 0$ and ending at time $t = 8$.

$$\mu_Y = \mu_X \int_{-\infty}^{\infty} h(t)dt = 8\mu_X$$

Squaring both sides, we see that $\mu_Y^2 = 384$.

Problem 4.20: In the block diagram

$$X(t) + Z(t) \to \boxed{H(\omega)} \to X_0(t) + Z_0(t)$$

the PSD's of WSS signal $X(t)$ and WSS noise $Z(t)$ are given by

$$S_X(\omega) = \begin{cases} 10 - |\omega|, & -10 \le \omega \le 10 \\ 0, & \text{elsewhere} \end{cases} \qquad S_Z(\omega) = \begin{cases} 5, & -10 \le \omega \le 10 \\ 0, & \text{elsewhere} \end{cases}$$

The filter is an ideal low-pass filter with frequency response $H(\omega)$ equal to 1 from $-B$ to B (zero elsewhere), where B, the filter bandwidth, is to be determined by you in the following.

(a) Compute the signal-to-noise ratio at the filter input (i.e., the $X(t)$ power divided by the $Z(t)$ power). (Hint: If you use the plots of $S_X(\omega)$ and $S_Z(\omega)$, you may be able to compute power without doing any integration.)

Solution.

$$\text{signal-to-noise ratio} \ = \frac{\text{area under } S_X(\omega)}{\text{area under } S_Z(\omega)} = \frac{100}{100} = 1$$

(b) $X_0(t)$ is the signal part of the filter output (i.e., the filter's response to $X(t)$) and $Z_0(t)$ is the noise part of the filter output. Compute the filter bandwidth B so that the filter output signal-to-noise ratio will be 1.75 times the filter input signal-to-noise ratio.

Solution. The output signal-to-noise ratio is

$$\frac{\int_{-B}^{B}(10 - |\omega|)d\omega}{\int_{-B}^{B} 5d\omega} = \frac{100 - (10 - B)^2}{10B} = 1.75$$

Solving this equation for B, you get $B = 2.5$. (I did not do any of the integrals above; I just used the formula for the area of a triangle.)

Problem 4.21: Two independent discrete-time ergodic WSS processes X_n and Y_n have autocorrelation functions

$$\begin{aligned} R_X(\tau) &= 40\delta[\tau] + 25 \\ R_Y(\tau) &= 70\delta[\tau] + 16 \end{aligned}$$

and it is known that μ_X and μ_Y are both positive. Let Z_n be the WSS process

$$Z_n = 3X_n - 2Y_n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

(a) Find μ_X, μ_Y, μ_Z.

Solution. We have

$$\mu_X^2 = 25, \quad \mu_Y^2 = 16,$$

and so

$$\mu_X = 5, \quad \mu_Y = 4.$$

Therefore,

$$\mu_Z = 3\mu_X - 2\mu_Y = 7.$$

(b) Find the power generated by the process Z_n.

Solution.

$$\begin{aligned} E[Z_n^2] &= 9E[X_n^2] + 4E[Y_n^2] - 12E[X_nY_n] \\ &= 9P_X + 4P_Y - 12\mu_X\mu_Y \\ &= 9(40+25) + 4(70+16) - 12*5*4 = 689 \end{aligned}$$

Problem 4.22: Let X be a discrete-time WSS process with autocorrelation function

$$R_X(\tau) = \exp(-|\tau|).$$

X is to be filtered to yield a discrete-time WSS filter output process Y for which $R_Y(1) = 0$. The filter impulse response function is to be of the form

$$h[n] = \delta[n] + a\delta[n-1].$$

Determine the value of the filter tap weight a.

Solution. Since

$$R_Y(\tau) = R_X(\tau) * (h[\tau] * h[-\tau]),$$

and since

$$h[\tau] * h[-\tau] = (1+a^2)\delta[\tau] + a\delta[\tau+1] + a\delta[\tau-1],$$

it follows that

$$R_Y(\tau) = (1+a^2)R_X(\tau) + aR_X(\tau+1) + aR_X(\tau-1).$$

Plugging in $\tau = 1$, we get

$$0 = R_Y(1) = (1+a^2)\exp(-1) + a(1+\exp(-2)).$$

Solving for a, you get

$$a = -\exp(-1).$$

Problem 4.23: Let a WSS process X(t) have PSD

$$S_X(\omega) = \frac{90\omega^2}{10+\omega^4}$$

(a) Write Matlab code which will compute the power P_X generated by the X process.

Solution.

```
syms omega
power=int((1/(2*pi))*90*omega^2/(10+omega^4),-inf,inf);
PX=eval(power)
```

(b) Suppose we pass the process $X(t)$ through an ideal lowpass filter with bandwidth 50 rad/sec, and let $Y(t)$ be the output process. Write Matlab code which computes the power P_Y generated by the Y process.

Solution.

```
syms omega
power=int((1/(2*pi))*90*omega^2/(10+omega^4),-50,50);
PY=eval(power)
```

(c) Use Matlab to find the bandwidth B in rad/sec of an ideal lowpass filter which passes 90% of X's power through it. Round your answer for B to nearest integer.

Solution. Run the following line of Matlab code

```
eval(int((1/(2*pi))*90*omega^2/(10+omega^4),-B,B))/PX
```

for each of the following B values:

$$B = 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20.$$

You will find one of these that does the job.

Problem 4.24: Let (Z_n) be white noise with variance 9. Let (X_n) be the WSS process

$$X_n = 5Z_n - 4Z_{n-1} + 2Z_{n-2}$$

Use Matlab to obtain $R_X(\tau)$ for $\tau = -2, -1, 0, 1, 2$.

Solution. Run

```
9*conv([5 -4 2],[2 -4 5])
```

This is an implementation of the formula

$$R_X(\tau) = R_Z(\tau) * h(\tau) * h(-\tau) = 9[h(\tau) * h(-\tau)].$$

Problem 4.25: Let Z be discrete-time white noise with unit variance. Find an FIR filter which converts Z into process X with autocorrelation function

$$R_X(\tau) = \begin{cases} 5/4, & \tau = 0 \\ -1/2, & \tau = \pm 1 \\ 0, & \text{elsewhere} \end{cases}$$

Solution. From experience we know that the filter transfer function is of the form

$$H(z) = a + bz^{-1}.$$

Computing $H(z)H(z^{-1})$ will give us $S_X(\omega)$ (after converting from the z variable to the ω variable).

$$H(z)H(z^{-1}) = (a^2 + b^2) + ab(z + z^{-1}).$$

This must coincide with the z-transform of $R_X(\tau)$, which by inspection is

$$5/4 - 1/2(z + z^{-1}).$$

Therefore, we must solve the equations

$$\begin{aligned} a^2 + b^2 &= 5/4 \\ ab &= -1/2 \end{aligned}$$

Simple algebra gives us two solutions:

$$a = 1, \quad b = -1/2,$$

or

$$a = -1, \quad b = 1/2,$$

Taking the first solution, we can take our FIR filter to have impulse response function

$$h[n] = a\delta[n] + b\delta[n-1] = \delta[n] - 0.5\delta[n-1].$$

5 Gaussian Processes

Problem 5.1: A WSS Gaussian process $X(t)$ has $\mu_X > 0$ and

$$R_X(\tau) = 4 + 5(2^{-|\tau|})$$

(a) Let $f(x)$ be the density of $X(1)$. Write down $f(x)$.

Solution.

$$\begin{aligned} E[X(1)]^2 &= \lim_{\tau\to\infty} R_X(\tau) = 4 \\ E[X(1)] &= 2 \\ E[X(1)^2] &= R_X(0) = 9 \\ \text{Var}[X(1)] &= E[X(1)^2] - E[X(1)]^2 = 9 - 4 = 5 \\ f(x) &= \frac{1}{\sqrt{10\pi}} \exp(-(x-2)^2/10) \end{aligned}$$

(b) Let $g(y)$ be the density of $Y = X(3) + X(2)$. Write down $g(y)$.

Solution.

$$\begin{aligned}
E[Y] &= 2\mu_X = 4 \\
E[Y^2] &= E[X(2)^2] + E[X(3)^2] + 2E[X(2)X(3)] \\
&= 2R_X(0) + 2R_X(1) = 18 + 2(4 + 5/2) = 31 \\
\sigma_Y^2 &= E[Y^2] - \mu_Y^2 = 31 - 16 = 15 \\
g(y) &= \frac{1}{\sqrt{30\pi}} \exp(-(y-4)^2/30)
\end{aligned}$$

Problem 5.2: $X[n] = A(-1)^n + B$ is the WSS Gaussian process in which A, B are independent standard Gaussian random variables.

(a) Find $R_X(\tau)$.

Solution.

$$\begin{aligned}
R_X(\tau) &= E[X[n]X[n+\tau]] \\
&= E[A^2](-1)^{2n+\tau} + E[B^2] + E[AB]((-1)^n + (-1)^{n+\tau}) \\
&= (-1)^\tau + 1
\end{aligned}$$

(The last term dropped out because $E[AB] = E[A]E[B] = 0$.)

(b) Determine the probability distribution of each 1-D cross-section of the process.

Solution. If n is odd, then $X[n] = B - A$, and this cross-section has a Gaussian distribution with mean 0 and variance 2. If n is even, then $X[n] = A + B$, which also has a Gaussian distribution with mean 0 and variance 2.

(c) Explain why the process is not ergodic.

Solution. Notice that

$$X[n]X[n+\tau] = A^2(-1)^\tau + B^2 + AB[(-1)^n][1 + (-1)^\tau].$$

Fixing τ, and time-averaging this expression over n, the last term drops out (since $(-1)^n$ oscillates between $+1$ and -1). However, the other two terms do not drop out, and so we conclude that the time-average of $X[n]X[n+\tau]$ is

$$A^2(-1)^\tau + B^2.$$

Since this depends on the realization, the process can't be ergodic.

(d) Compute $E[X[n]^4]$.

Solution. From solution to (b), $X[n]/\sqrt{2}$ is standard Gaussian. The fourth moment of a standard Gaussian random variable is 3. (You can obtain this by differentiating the moment generating function $e^{s^2/2}$ four times and then plugging in $s = 0$.) That is,

$$E[(X[n]/\sqrt{2})^4] = 3.$$

It follows that

$$E[X[n]^4] = 12.$$

Problem 5.3: An ergodic WSS Gaussian process

$$X(t), \quad -\infty < t < \infty$$

has autocorrelation function

$$R_X(\tau) = 100e^{-\tau^2}\cos(2\pi\tau) + 10\cos(6\pi\tau) + 36.$$

(a) Find the constant value of $E[X(t)]$ (assuming that this constant value is positive).

Solution. Since the process is ergodic, you can compute the following time average to get μ_X^2:

$$\mu_X^2 = \lim_{T\to\infty} T^{-1}\int_0^T R_X(\tau)d\tau.$$

There are three terms in $R_X(\tau)$ which we can time average separately:

- The time average of $100e^{-\tau^2}\cos(2\pi\tau)$ is zero because this is the limit as $\tau \to \infty$.
- The time average of $10\cos(6\pi\tau)$ is zero because the time average of any sinusoid is zero.
- The time average of 36 is 36.

Our conclusion is that

$$\mu_X^2 = 36.$$

Therefore, $\mu_X = \pm 6$. We are told that $\mu_X > 0$, and so

$$E[X(t)] = \mu_X = 6.$$

(b) Find the constant value of $E[X(t)^2]$.

$$E[X(t)^2] = P_X = R_X(0) = 146.$$

(c) Find the constant value of $Var[X(t)]$.

Solution.

$$Var[X(t)] = \sigma_X^2 = R_X(0) - \mu_X^2 = 110.$$

(d) Find the smallest positive value of τ for which the random variables $X(t)$ and $X(t+\tau)$ are uncorrelated.

Solution. Uncorrelated means (see Chapters 3,5)

$$E[X(t)X(t+\tau)] = E[X(t)]E[X(t+\tau)].$$

The left side is $R_X(\tau)$, and the right side is $\mu_X^2 = 36$. So, we find the desired τ by solving the equation

$$100e^{-\tau^2}\cos(2\pi\tau) + 10\cos(6\pi\tau) = 0. \tag{4}$$

One solution that can be seen by inspection is $\tau = 0.25$. If one uses Matlab to plot the function of τ on the left side of equation (4), one sees that $\tau = 0.25$ is the smallest positive value of τ which makes this function equal to zero. So, the answer is

$$\tau = 0.25.$$

(e) What is the mean and variance of the random variable $Y = X(0) + X(1)$? What is $P(Y \leq 40)$?

$$\mu_Y = E[X(0)] + E[X(1)] = 2 * \mu_X = 12.$$
$$Cov(X(0), X(1)) = E[X(0)X(1)] - \mu_X^2 = R_X(1) - 36 = 46.7879.$$
$$\sigma_Y^2 = Var[X(0)] + Var[X(1)] + 2Cov(X(0), X(1)) = 313.5759.$$
$$P(Y \leq 40) = P\left(\frac{Y - \mu_Y}{\sigma_Y} \leq \frac{28}{\sqrt{313.5759}}\right) = \Phi(1.5812) = 0.94.$$

6 Independent Increments Processes

Problem 6.1: Two famous "independent increments processes" are the random walk process and the Poisson process. This problem concerns these two processes.

(a) Let

$$X_n, \quad n = 0, 1, 2, 3, \cdots$$

be the random walk process. (Recall that $X_0 = 0$, and that each X_n for $n > 1$ is the sum of the first n samples of the Bernoulli fair-coin-flipping process.) Compute the probability that the random walk process makes its first return to zero at time 6. In other words, compute

$$P[X_1 \neq 0, X_2 \neq 0, X_3 \neq 0, X_4 \neq 0, X_5 \neq 0, X_6 = 0].$$

Solution. The "increments" are:

$$\begin{aligned} B_1 &= X_1 - X_0 \\ B_2 &= X_2 - X_1 \\ B_3 &= X_3 - X_2 \\ B_4 &= X_4 - X_3 \\ B_5 &= X_5 - X_4 \\ B_6 &= X_6 - X_5 \end{aligned}$$

The sequence of increments $(B_1, B_2, B_3, B_4, B_5, B_6)$ must be one of the following:

$$(+1, +1, +1, -1, -1, -1)$$
$$(-1, -1, -1, +1, +1, +1)$$
$$(+1, +1, -1, +1, -1, -1)$$
$$(-1, -1, +1, -1, +1, +1)$$

By "independent increments" each of these 4 possibilities has probability $(1/2)^6$. The answer is therefore

$$4 * (1/2)^6 = 1/16.$$

(b) Let

$$X(t), \quad t \geq 0$$

be the Poisson process with arrival rate of 0.5 arrivals per second (our time variable t is in seconds). Compute the probability that there are just as many arrivals in the first three seconds as there are in the next four seconds. In other words, compute

$$P[X(7) = 2X(3)].$$

Solution. The desired probability can be decomposed as the following sum:

$$\sum_{k=0}^{\infty} P[X(3) - X(0) = k, X(7) - X(3) = k].$$

By "independent increments", the k-th term can be re-written as:

$$P[X(3) - X(0) = k]P[X(7) - X(3) = k].$$

$X(3) - X(0)$ is a Poisson RV with parameter

$$\alpha = (3 - 0) * 0.5 = 1.5.$$

Therefore

$$P[X(3) - X(0) = k] = \exp(-1.5)(1.5)^k/k!.$$

$X(7) - X(3)$ is a Poisson RV with parameter

$$\alpha = (7 - 3) * 0.5 = 2.$$

Therefore

$$P[X(7) - X(3) = k] = \exp(-2)2^k/k!.$$

Our answer is therefore

$$\exp(-3.5) \sum_{k=0}^{\infty} \frac{3^k}{(k!)^2} = 0.2162.$$

Problem 6.2: $X(t)$ is Poisson process with arrival rate $\lambda = 4$.

(a) Compute $P[X(1) = 4]$ and $P[X(2) = 2]$.
Solution. The parameter of the Poisson RV $X(1)$ is 4. Therefore:

$$P[X(1) = 4] = \exp(-4) * 4^4/4! = 0.1954$$

The parameter of the Poisson RV $X(2)$ is 8. Therefore:

$$P[X(2) = 2] = \exp(-8) * 8^2/2 = 0.0107$$

(b) Compute $P[X(2) > X(1) + 1]$.

Solution.

$$P[X(2) > X(1) + 1] = P[X(2) - X(1) > 1] = 1 - P[X(2) - X(1) \leq 1]$$

$X(2) - X(1)$ is Poisson with parameter 4. Therefore:

$$P[X(2) > X(1) + 1] = 1 - \exp(-4) - \exp(-4) * 4 = 0.9084$$

(c) Compute $P[X(3) > X(2) > X(1)]$.

Solution. Since the increments $X(3) - X(2)$ and $X(2) - X(1)$ are independent,

$$\begin{aligned} P[X(3) > X(2) > X(1)] &= P[X(3) - X(2) > 0,\ \ X(2) - X(1) > 0] \\ &= P[X(3) - X(2) > 0]P[X(2) - X(1) > 0] \end{aligned}$$

Both $X(3) - X(2)$ and $X(2) - X(1)$ are Poisson with parameter 4. Therefore:

$$P[X(3) > X(2) > X(1)] = (1 - \exp(-4))^2 = 0.9637$$

(d) Compute $E[X(2)|X(1) = 5]$.

Solution. $X(2) - X(1)$ and $X(1)$ are independent. Therefore:

$$\begin{aligned} E[X(2)|X(1) = 5] &= E[X(2) - X(1)|X(1) = 5] + E[X(1)|X(1) = 5] \\ &= E[X(2) - X(1)] + 5 \\ &= 4 + 5 = 9 \end{aligned}$$

(e) Compute $E[X(2)X(5)]$.

Solution. Using independent increments property of Poisson process again,

$$\begin{aligned} E[X(2)X(5)] &= E[X(2)^2] + E[X(2)(X(5) - X(2)] \\ &= E[X(2)^2] + E[X(2)]E[X(5) - X(2)] \end{aligned}$$

$X(2)$ is Poisson with parameter 8, mean 8, variance 8, and second moment $8+8^2 = 72$. $X(5) - X(2)$ is Poisson with parameter 12, mean 12. Therefore:

$$E[X(2)X(5)] = 72 + 8 * 12 = 168$$

Problem 6.3: Let $W(t)$ be a Brownian motion process with parameter $\alpha = 1$. Compute

(a) The correlation coefficient between $W(3)$ and $W(9)$, using independent increments property.

Solution. The mean function of the Brownian motion process is zero. Also,

$$E[W(t)^2] = Var(W(t)) = \alpha t = t,$$

for all $t \geq 0$.

Therefore

$$Cov(W(3), W(9)) = E[W(3)W(9)].$$

We have

$$W(3)W(9) = W(3)[W(9) - W(3) + W(3)] = W(3)[W(9) - W(3)] + W(3)^2.$$

By independent increments, $W(9) - W(3)$ and $W(3) = W(3) - W(0)$ are independent; they also each have zero mean. Therefore,

$$E\{W(3)[W(9) - W(3)]\} = E[W(3)]E[W(9) - W(3)] = 0,$$

and so

$$E[W(3)W(9)] = E\{W(3)[W(9) - W(3)]\} + E[W(3)^2] = E[W(3)^2] = 3.$$

We conclude that

$$\rho_{W(3),W(9)} = \frac{Cov(W(3), W(9))}{\sqrt{Var(W(3))}\sqrt{Var(W(9))}} = \frac{3}{\sqrt{3}\sqrt{9}} = \frac{1}{\sqrt{3}}$$

(b) Compute $E[(W(1) + W(2) + W(3))^2]$ using independent increments.

Solution.

$$W(1)+W(2)+W(3) = W(1)+2W(2)+[W(3)-W(2)] = 3W(1)+2[W(2)-W(1)]+[W(3)-W(2)].$$

Also,

$$W(1) = W(1) - W(0),$$

so we have expressed $W(1)+W(2)+W(3)$ as a linear combination of three increments:

$$W(1) + W(2) + W(3) = 3[W(1) - W(0)] + 2[W(2) - W(1)] + [W(3) - W(2)],$$

Each of the three increments $W(1) - W(0)$, $W(2) - W(1)$, and $W(3) - W(2)$ is a Gaussian(0,1) RV, and these three RV's are independent. It follows that

$$E[(W(1)+W(2)+W(3))^2] = 3^2E[(W(1)-W(0)^2]+2^2E[(W(2)-W(1))^2]+E[(W(3)-W(2))^2].$$

(All cross-product terms on right side drop out because of independent increments, that is, for $i \neq j$,

$$E[(W(i)-W(i-1))(W(j)-W(j-1))] = E[W(i)-W(i-1)]E[W(j)-W(j-1)] = 0*0 = 0$$

holds.) We conclude that

$$E[(W(1) + W(2) + W(3))^2] = 3^2 + 2^2 + 1 = 14.$$