6.431 Quiz 1 Solutions
October 9, 2002

Problem 1: (52 points)

(a) (12 pts) First we find the probability q of the event "more of the stocks held by Bill are up than down". Since the n stocks are independent and go up and down in a statistically identical way, we can view the experiment *each day* as n IID flips of a coin with probability of success p. The number of successes is hence a Binomial random variable with parameters n and p.

$$q = \sum_{k \geq \lceil n/2 \rceil}^{n} \binom{n}{k} p^k (1-p)^{n-k} > 0.$$

If the experiment is repeated independently every day, then q may be interpreted as a frequency of occurrence of the event. Hence, intuitively this event will occur infinitely often. To make this statement exact, we apply the Borel Cantelli lemma: $\sum q = \infty$ and the experiments are independent, hence with probability one, the event will occur infinitely often.

(b) (12 pts) Assume that stocks $1, \ldots, 2m$ get doubled or divided by 2, stocks $2m+1, \ldots, 4m$ the ones that get tripled or divided by 3, and $4m+1, \ldots, 6m$ the ones that get quadrupled or divided by 4. Let X_j be the random variable representing the value Bill holds in stock j after one day. Using the fact that expectation is linear and the fact that stocks go up and down in an IID manner,

$$\begin{aligned}
E[X_1 + \cdots + X_n] &= \sum_{j=1}^{2m} E[X_j] + \sum_{j=2m+1}^{4m} E[X_j] + \sum_{j=4m+1}^{6m} E[X_j] \\
&= 2mE[X_1] + 2mE[X_{2m+1}] + 2mE[X_{4m+1}] \\
&= 2m\left(\frac{2}{n}p + \frac{1}{2n}(1-p)\right) + 2m\left(\frac{3}{n}p + \frac{1}{3n}(1-p)\right) + 2m\left(\frac{4}{n}p + \frac{1}{4n}(1-p)\right) \\
&= \left(\frac{2}{3}p + \frac{1}{2*3}(1-p)\right) + \left(\frac{3}{3}p + \frac{1}{3*3}(1-p)\right) + \left(\frac{4}{3}p + \frac{1}{4*3}(1-p)\right) \\
&= 3p + \frac{13}{36}(1-p)
\end{aligned}$$

(c) (14 pts) Given that $n/2$ stocks were up, all possible choices are equally likely, and hence, to compute the probability of the event that they included the high gainers we count the favorable outcomes and divide by the number of total possible outcomes.

From another perspective the experiment can be modeled as choosing $n/2$ balls out of n balls a third of which is red, a third is blue and a third is green, and we would like to compute the probability of selecting all the red ones.

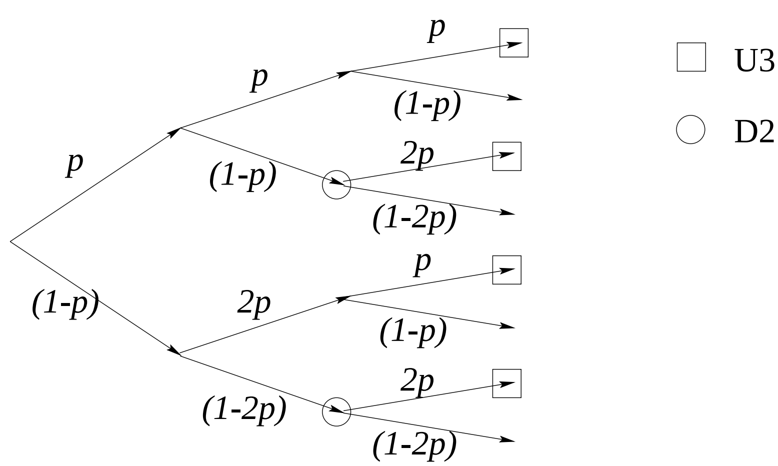
The total number of choices is: $\binom{n}{n/2}$. Since all the red balls have to be chosen, the number of favorable outcomes is equal to the number of ways we can choose the rest: $\binom{2n/3}{n/2-n/3}$. Therefore, the probability of the event is

$$\frac{\binom{4m}{m}}{\binom{6m}{3m}}.$$

(d) (14 pts) Given that EECS went up on the third day, the probability that Bill invested more money in the stock is that of EECS going down on the second day. Denoting these events by $U3$ and $D2$ respectively we need to compute the probability $P(D2|U3)$ for which we use Bayes' rule:

$$\begin{aligned} P(D2|U3) &= \frac{P(U3|D2)P(D2)}{P(U3)} \\ &= \frac{2pP(D2)}{P(U3)} \end{aligned}$$

It remains to compute $P(D2)$ and $P(U3)$ which can be readily obtained by drawing a tree



Hence:

$$\begin{aligned} P(D2) &= p(1-p) + (1-p)(1-2p) \\ P(U2) &= pp + (1-p)2p \\ P(U3) &= 2pP(D2) + pP(U2) \end{aligned}$$

In summary:

$$\begin{aligned} P(D2|U3) &= \frac{2p(1-p)^2}{2p(1-p)^2 + p^2(2-p)} \\ &= \frac{2(1-p)^2}{2(1-p)^2 + p(2-p)} \end{aligned}$$

Problem 2: (46 points) Ramzi arrives first and always parks at the edge of a single row of n spaces and Danielle arrives later and always parks as close to Ramzi as possible.

(a) (16 pts) After Ramzi parks, there are $n-1$ consecutive spaces remaining. (We assume that $k < n-1$ which implies both that all k cars and Danielle will always find a spot.) The total number of ways k people can park in these $n-1$ remaining spots is $\binom{n-1}{k}$. Note that the closest open spot to Ramzi when Danielle arrives can be no further than k spots away, corresponding to the case that the k cars choose to fill the k closest spots to Ramzi i.e.,

$x \in \{0, 1, \dots k\}$. If the closest open spot to Ramzi when Danielle arrives is x spots away, then x of the k cars must have parked in between Danielle and Ramzi and the remaining $k - x$ have *not* chosen the spot left open for Danielle. In other words, the number of ways the k cars can arrange themselves such that Danielle will end up x spots away from Ramzi is $\binom{n-2-x}{k-x}$. Hence, the PMF for random variable X is

$$p_X(x) = \frac{\binom{n-2-x}{k-x}}{\binom{n-1}{k}}, \quad x \in \{0, 1, \dots k\} \quad .$$

(b) (15 pts) Let us consider the outcome of any day a "success" provided Danielle is able to park two spots or less from Ramzi, occurring with probability $p = \mathbf{P}(X \leq 2) = p_0 + p_1 + p_2$; otherwise, the outcome of any day is a "failure," occurring with probability $1 - p$. We are interested in the expected number of days *between two successes* of a sequence of independent Bernoulli trials; stated otherwise, given we just experienced a success, what is the expected number of trials between now and the next success. Thus, we are asking for the expected value of a geometric random variable Z with parameter p, describing the number of trials *up to and including the first success*, and subtracting one because we do not wish to include in our count the trial corresponding to the awaited success:

$$\mathbf{E}[Z-1] = \frac{1}{p} - 1 = \frac{1-p}{p} = \frac{1 - p_o - p_1 - p_2}{p_0 + p_1 + p_2} \quad .$$

(c) (15 pts) Again assuming independence on consecutive days, we have

$$\begin{aligned} \text{var}(Y) &= \text{var}\left(\frac{1}{m}\sum_{j=1}^{m} X_j\right) = \frac{1}{m^2}\left[\text{var}(X_1 + X_2 + \cdots + X_m)\right] \\ &= \frac{1}{m^2}\left[\text{var}(X_1) + \text{var}(X_2) + \cdots + \text{var}(X_m)\right] = \frac{1}{m^2}\left[m \cdot \text{var}(X)\right] = \frac{1}{m}\text{var}(X) \quad . \end{aligned}$$

In terms of the p_i's,

$$\text{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \sum_{i=0}^{k} i^2 p_i - \left(\sum_{i=0}^{k} i p_i\right)^2$$

so

$$\text{var}(Y) = \frac{1}{m}\left(\sum_{i=0}^{k} i^2 p_i - \left(\sum_{i=0}^{k} i p_i\right)^2\right) \quad .$$