Massachusetts Institute of Technology

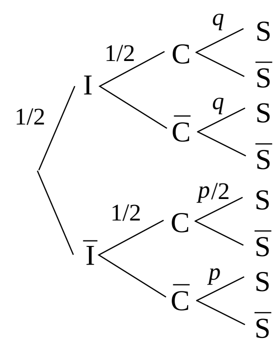
Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2002)

6.041 Quiz 1 Solutions

Problem 1: (48 points) Any one morning's outcome for Sue is characterized by the following probability tree, where unlabelled branch probabilities are implied as they correspond to complementary events of the labelled braches:



Here, events I, C and S correspond to "Sue parks inside," "the night is cold" and "Sue's car starts," respectively.

(a) (16 pts) Bayes' rule says $\mathbf{P}(I \mid S) = \mathbf{P}(S \mid I)\mathbf{P}(I)/\mathbf{P}(S) = q \cdot \frac{1}{2}/\mathbf{P}(S)$ and via the Total Probability Theorem

$$\mathbf{P}(S) = \left(q \cdot \frac{1}{2} + q \cdot \frac{1}{2}\right)\frac{1}{2} + \left(\frac{p}{2} \cdot \frac{1}{2} + p \cdot \frac{1}{2}\right)\frac{1}{2} = \frac{4q+3p}{8} \quad .$$

(b) (16 pts) The new information corresponds to changing the probability on the branch between $\bar{I}$ and C from $\frac{1}{2}$ to $\frac{4}{5}$. We have $\mathbf{P}(C \mid S) = \mathbf{P}(S \mid C)\mathbf{P}(C)/\mathbf{P}(S)$ and using similar calculations as in part (a):

$$\begin{aligned}
\mathbf{P}(S \mid C) &= q \cdot \frac{1}{2} + \frac{p}{2} \cdot \frac{1}{2} = \frac{2q+p}{4} \quad , \\
\mathbf{P}(C) &= \frac{1}{2} \cdot \frac{1}{2} + \frac{4}{5} \cdot \frac{1}{2} = \frac{13}{20} \quad , \qquad \text{and} \\
\mathbf{P}(S) &= \left(q \cdot \frac{1}{2} + q \cdot \frac{1}{2}\right)\frac{1}{2} + \left(\frac{p}{2} \cdot \frac{4}{5} + p \cdot \frac{1}{5}\right)\frac{1}{2} = \frac{5q+3p}{10} \quad .
\end{aligned}$$

(c) (16 pts) We are given that Kate's car behaves like Sue's car, in the sense that its probability of starting is also affected by the weather and thus also by being parked inside or outside. However, only one of these cars on any given day can be inside and the other must be outside. For example, assuming $p \leq q$, given the event "Sue's car started" leads us to believe it is more likely that Sue's car is inside and, hence, more likely that Kate's car is outside that in turn leads us to believe that the event "Kate's car started" is less likely. In general, because the event "Sue's car starts" impacts the probability of the event "Kate parks inside," it also impacts the probability of the event "Kate's car starts." Mathematically, letting event K denote "Kate's car started," we have just argued that

$$\mathbf{P}(K \mid S) \neq \mathbf{P}(K) \quad .$$

We conclude the two events are *dependent*.

Problem 2: (50 points) Ramzi arrives first and always parks at the edge of a single row of n spaces and Danielle arrives later and always parks as close to Ramzi as possible.

(a) (18 pts) After Ramzi parks, there are $n-1$ consecutive spaces remaining. (We assume that $k < n-1$ which implies both that all k cars and Danielle will always find a spot.) The total number of ways k people can park in these $n-1$ remaining spots is $\binom{n-1}{k}$. Note that the closest open spot to Ramzi when Danielle arrives can be no further than k spots away, corresponding to the case that the k cars choose to fill the k closest spots to Ramzi i.e., $x \in \{0, 1, \ldots k\}$. If the closest open spot to Ramzi when Danielle arrives is x spots away, then x of the k cars must have parked inbetween Danielle and Ramzi and the remining $k-x$ have *not* chosen the spot left open for Danielle. In other words, the number of ways the k cars can arrange themselves such that Danielle will end up x spots away from Ramzi is $\binom{n-2-x}{k-x}$. Hence, the PMF for random variable X is

$$p_X(x) = \frac{\binom{n-2-x}{k-x}}{\binom{n-1}{k}}, \quad x \in \{0, 1, \ldots k\} \qquad .$$

(b) (16 pts) Let us consider the outcome of any day a "success" provided Danielle is able to park two spots or less from Ramzi, occurring with probability $p = \mathbf{P}(X \leq 2) = p_0 + p_1 + p_2$; otherwise, the outcome of any day is a "failure," occurring with probability $1-p$. We are interested in the expected number of days *between two successes* of a sequence of independent Bernoulli trials; stated otherwise, given we just experienced a success, what is the expected number of trials between now and the next success. Thus, we are asking for the expected value of a geometric random variable Z with parameter p, describing the number of trials *up to and including the first success*, and subtracting one because we do not wish to include in our count the trial corresponding to the awaited success:

$$\mathbf{E}[Z-1] = \frac{1}{p} - 1 = \frac{1-p}{p} = \frac{1 - p_o - p_1 - p_2}{p_0 + p_1 + p_2} \qquad .$$

(c) (16 pts) Again assuming independence on consecutive days, we have

$$\begin{aligned} \text{var}(Y) &= \text{var}\left(\frac{1}{m}\sum_{j=1}^{m} X_j\right) = \frac{1}{m^2}\left[\text{var}(X_1 + X_2 + \cdots + X_m)\right] \\ &= \frac{1}{m^2}\left[\text{var}(X_1) + \text{var}(X_2) + \cdots + \text{var}(X_m)\right] = \frac{1}{m^2}\left[m \cdot \text{var}(X)\right] = \frac{1}{m}\text{var}(X) \qquad . \end{aligned}$$

In terms of the p_i's,

$$\text{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \sum_{i=0}^{k} i^2 p_i - \left(\sum_{i=0}^{k} i p_i\right)^2$$

so

$$\text{var}(Y) = \frac{1}{m}\left(\sum_{i=0}^{k} i^2 p_i - \left(\sum_{i=0}^{k} i p_i\right)^2\right) \qquad .$$