MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2002)

Quiz 1 Review Solutions
October 8, 2002

1. Consider a sequence of six independent rolls of this die, and let x_i be the random variable corresponding to the ith roll.

   (a) What is the probability that exactly three of the rolls have an outcome equal to three? Each roll x_i can either be a three with probability 1/4 or not a three with probability 3/4. There are $\binom{6}{3}$ ways of placing the threes in the sequence of 6 rolls. After we require that a three go in each of these spots, which takes probablity $\frac{1}{4}^3$ our only remaining condition is that either a one or a two go in the other three spots, which takes probability $\frac{3}{4}^3$. So the probability of exactly three rolls in a sequence of 6 independent rolls is $\boxed{\binom{6}{3}(\frac{1}{4})^3(\frac{3}{4})^3}$.

   (b) What is the probability that the first roll is a 1, given that exactly two of the six rolls had an outcome of 1? The probability of obtaining a one on a single roll is 1/2, and the probability of obtaining a 2 or 3 on a single roll is also 1/2. For the purposes of solving this problem we treat obtaining a 2 or 3 as an equivalent outcome. We know that there are $\binom{6}{2}$ ways of rolling exactly 2 ones. Of these $\binom{6}{2}$ ways exactly $\binom{5}{1} = 5$ ways result in a one in the first roll, since we can place the remaining one in any of the 5 remaining rolls. The rest of the rolls must be either two or three. Thus the probability that the first roll is a one given exactly 2 rolls had an outcome of one is $\boxed{\frac{5}{\binom{6}{2}}}$.

   (c) We are now told that exactly three of the rolls resulted in one and exactly three resulted in 2. What is the probability of the outcome 121212? We want to find

   $$P(121212 \mid exactly\ 3\ ones\ and\ 3\ twos) = \frac{P(121212)}{P(exactly\ 3\ ones\ and\ 3\ twos)}.$$

   Any particular sequence of three ones and three twos will have the same probability: $\frac{1}{2}^3\frac{1}{4}^3$. There are $\binom{6}{3}$ possible rolls with exactly three ones and three twos.
   Therefore $P(121212 \mid exactly\ 3\ ones\ and\ 3\ twos) = \boxed{\frac{1}{\binom{6}{3}}}$.

   (d) Conditioned on the event that at least one roll resulted in 3, find the conditional PMF of the number of 3's. Let A be the event that at least one roll results in a three. Then P(A) = 1 - P(no rolls resulted in three) $= 1 - (\frac{3}{4})^6$. Now let k be the random variable representing the number of threes in the 6 rolls. Our unconditional PMF $P_K(k)$ for k is given by

   $$P_K(k) = \binom{6}{k}\frac{1}{4}^k\frac{3}{4}^{6-k}$$

   We find the conditional PMF $P_{k|A}(k|A)$ for k using the definition of conditional probability:

   $$P_{K|A}(k|A) = \frac{P(K=k,A)}{P(A)}$$

   Thus we obtain

   $$P_{K|A}(k|A) = \begin{cases} \frac{1}{1-(3/4)^6}\binom{6}{k}(\frac{1}{4})^k(\frac{3}{4})^{6-k} & for\ k = 1, 2, \ldots, 6, \\ 0 & otherwise \end{cases}$$

   Note that $P_{K|A}(0|A) = 0$ because the event $k = 0$ and the event A are mutually exclusive. Thus the probability of their intersection, which appears in the numerator in the definition of the conditional PMF, is zero.

2. Suppose the president decides to investigate A first. Then her expected costs will be

$$E(\text{costs}) = D_A + pR_A + (1-p)\cdot(D_B + R_B),$$

whereas if she investigates B first, then

$$E(\text{costs}) = D_B + (1-p)\cdot R_B + p\cdot(D_A + R_A).$$

In order that the first be smaller than the second, we need

$$pD_B > (1-p)D_A.$$

3. (a) Use the total probability theorem by conditioning on the number of questions that May has to answer. Let A be the event that she gives all wrong answers in a given lecture, let B_1 be the event that she gets one question in a given lecture, and let B_2 be the event that she gets two questions in a given lecture. Then:

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2)$$

From the problem statement, she is equally likely to get one or two questions in a given lecture, so $P(B_1) = P(B_2) = \frac{1}{2}$. Also, from the problem, $P(A|B_1) = \frac{1}{4}$, and, because of independence, $P(A|B_2) = (\frac{1}{4})^2 = \frac{1}{16}$. Thus overall:

$$P(A) = \frac{1}{4}\frac{1}{2} + \frac{1}{16}\frac{1}{2} = \frac{5}{32}.$$

(b) Let events A and B_2 be defined as in the previous part. Using Bayes' Rule:

$$P(B_2|A) = \frac{P(A|B_2)P(B_2)}{P(A)}$$

From the previous part, we said $P(B_2) = \frac{1}{2}$, $P(A|B_2) = \frac{1}{16}$, and $P(A) = \frac{5}{32}$. Thus;

$$P(B_2|A) = \frac{\frac{1}{16}\frac{1}{2}}{\frac{5}{32}} = \frac{1}{5}$$

As one would expect, given that May answers all the questions in a given lecture, it's more likely that she got only one question rather than two.

(c) We start by finding the PMF for X and Y. $p_X(x)$ is given from the problem statement:

$$p_X(x) = \begin{cases} \frac{1}{2}, & x = 1,2 \\ 0, & \text{otherwise} \end{cases}$$

The PMF for Y can be found by conditioning on X for each value that Y can take on. Because May can be asked at most two questions in any lecture, the range of Y is from 0 to 2. Thus for each value of Y, we find:

$$p_Y(0) = P(Y=0|X=0)P(X=1) + P(Y=0|X=2)P(X=2) = \frac{1}{4}\frac{1}{2} + \frac{1}{16}\frac{1}{2} = \frac{5}{32}$$

$$p_Y(1) = P(Y=1|X=1)P(X=1) + P(Y=1|X=2)P(X=2) = \frac{3}{4}\frac{1}{2} + 2\frac{3}{4}\frac{1}{4}\frac{1}{2} = \frac{9}{16}$$

$$p_Y(2) = P(Y=2|X=1)P(X=1) + P(Y=2|X=2)P(X=2) = 0\frac{1}{2} + (\frac{3}{4})^2\frac{1}{2} = \frac{9}{32}$$

Note that when calculating $P(Y=1|X=2)$, we got $2\frac{3}{4}\frac{1}{4}$ because there are two ways for May to answer one question right when she's asked two questions: either she answers the first question correctly or she answers the second question correctly. Thus, overall:

$$p_Y(y) = \begin{cases} 5/32, & y=0 \\ 9/16, & y=1 \\ 9/32, & y=2 \\ 0, & \text{otherwise} \end{cases}$$

Now the mean and variance can be calculated explicitly from the PMFs:

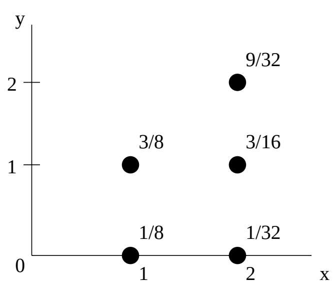
$$E[X] = 1\frac{1}{2} + 2\frac{1}{2} = \frac{3}{2}$$

$$var(X) = (1-\frac{3}{2})^2\frac{1}{2} + (2-\frac{3}{2})^2\frac{1}{2} = \frac{1}{4}$$

$$E[Y] = 0\frac{5}{32} + 1\frac{9}{16} + 2\frac{9}{32} = \frac{9}{8}$$

$$var(Y) = (0-\frac{9}{8})^2\frac{5}{32} + (1-\frac{9}{8})^2\frac{9}{16} + (2-\frac{9}{8})^2\frac{9}{32} = \frac{27}{64}$$

(d) The joint PMF $p_{X,Y}(x,y)$ is plotted below. There are only five possible (x,y) pairs. For each point, $p_{X,Y}(x,y)$ was calculated by $p_{X,Y}(x,y) = p_X(x)p_{Y|X=x}(y|X=x)$.



(e) By linearity of expectations,

$$E[Z] = E[X+2Y] = E[X] + 2E[Y] = \frac{3}{2} + 2\frac{9}{8} = \frac{15}{4}$$

Calculating $var(Z)$ is a little bit more tricky because X and Y are not independent; therefore we *cannot* add the variance of X to the variance of $2Y$ to obtain the variance of Z. (X and Y are clearly not independent because if we are told, for example, that $X=1$, then we know that Y cannot equal 2, although normally without any information about X, Y could equal 2.)
To calculate $var(Z)$, first calculate the PMF for Z from the joint PDF for X and Y. For each (x,y) pair, we assign a value of Z. Then for each value z of Z, we calculate $p_Z(z)$ by summing over the probabilities of all (x,y) pairs that map to z. Thus we get:

$$p_Z(z) = \begin{cases} 1/8, & z=1 \\ 1/32, & z=2 \\ 3/8, & z=3 \\ 3/16, & z=4 \\ 9/32, & z=6 \\ 0, & \text{otherwise} \end{cases}$$

In this example, each (x,y) mapped to exactly one value of Z, but this does not have to be the case in general. Now the variance can be calculated as:

$$var(Z) = \frac{1}{8}(1-\frac{15}{4})^2 + \frac{1}{32}(2-\frac{15}{4})^2 + \frac{3}{8}(3-\frac{15}{4})^2 + \frac{3}{16}(4-\frac{15}{4}) + \frac{9}{32}(6-\frac{15}{4})^2 = \frac{43}{16}$$

(f) For each lecture i, let Z_i be the random variable associated with the number of questions May gets asked plus two times the number May gets right. Also, for each lecture i, let D_i be the random variable $1000 + 40Z_i$. Let S be her semesterly salary. Because she teaches a total of 20 lectures, we have:

$$S = \sum_{i=1}^{20} D_i = \sum_{i=1}^{20} 1000 + 40Z_i = 20000 + 40\sum_{i=1}^{20} Z_i$$

By linearity of expectations,

$$E[S] = 20000 + 40E[\sum_{i=1}^{20} Z_i] = 20000 + 40(20)E[Z_i] = 23000$$

Since each of the D_i are independent, we have:

$$var(S) = \sum_{i=1}^{20} var(D_i) = 20var(D_i) = 20(var(1000 + 40Z_i)) = 20(40^2 var(Z_i)) = 36000$$

(g) Let Y be the expected number of questions she will answer wrong in a randomly chosen lecture. We can find $E[Y]$ by conditioning on whether the lecture is in math or in science. Let M be the event that the lecture is in math, and let S be the event that the lecture is in science. Then:

$$E[Y] = E[Y|M]P(M) + E[Y|S]P(S)$$

Since there are an equal number of math and science lectures and we are choosing randomly among them, $P(M) = P(S) = \frac{1}{2}$. Now we need to calculate $E[Y|M]$ and $E[Y|S]$ by finding the respective conditional PMFs first. The PMFs can be determined in an manner analagous to how we calculated the PMF for the number of correct answers in part c.)

$$p_{Y|S}(y|S) = \begin{cases} 5/32, & y = 2 \\ 9/16, & y = 1 \\ 9/32, & y = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$p_{Y|M}(y|M) = \begin{cases} \frac{1}{2}\frac{1}{10} + \frac{1}{2}(\frac{1}{10})^2 = 11/200, & y = 2 \\ \frac{1}{2}\frac{9}{10} + \frac{1}{2}2\frac{9}{10}\frac{1}{10} = 27/50, & y = 1 \\ \frac{1}{2}0 + \frac{1}{2}(\frac{9}{10})^2 = 81/200, & y = 0 \\ 0, & \text{otherwise} \end{cases}$$

Therefore:

$$E[Y|S] = 2\frac{5}{32} + 1\frac{9}{16} + 0\frac{9}{32} = \frac{7}{8}$$

$$E[Y|M] = 2\frac{11}{200} + 1\frac{27}{50} + 0\frac{81}{200} = \frac{13}{20}$$

This implies that:

$$E[Y] = \frac{13}{20}\frac{1}{2} + \frac{7}{8}\frac{1}{2} = \frac{61}{80}$$