6.041 Fall 2002 Quiz 2

50 minutes

DO NOT TURN THIS QUIZ OVER UNTIL YOU ARE TOLD TO DO SO

- You have 50 minutes to complete the quiz.
- Write your solutions in the exam booklet. We will not consider any work not in the exam booklet.
- This quiz has 2 problems, not necessarily in order of difficulty, that both relate to a common probabilistic situation.
- You may give an answer in the form of an arithmetic expression (sums, products, ratios, factorials) of numbers that could be evaluated using a calculator. Expressions like $\binom{8}{3}$ or $\sum_{k=0}^{5}(1/2)^k$ are also fine.
- A correct answer does not guarantee full credit and a wrong answer does not guarantee loss of credit. You should concisely indicate your reasoning and show all relevant work. The grade on each problem is based on our judgment of your level of understanding as reflected by what you have written.
- This is a closed-book exam except for two double-sided, handwritten, 8.5x11 formula sheets plus a calculator.
- Be neat! If we can't read it, we can't grade it.
- At the end of the quiz, turn in your solutions along with this quiz (this piece of paper).

Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2002)

Write your name and your TA's name on the front of the booklet. (2 points)

Both problems in this quiz correspond to the following probabilistic situation: Let X denote a zero-mean Gaussian random variable with variance $\sigma_X^2 > 0$. Consider the sequence of random variables $Y_0, Y_1, Y_2, \ldots$ defined by

$$\begin{aligned} Y_0 &= X \\ Y_k &= \alpha Y_{k-1} + V_k\,, \quad k = 1, 2, \ldots \end{aligned}$$

where α denotes an arbitrary constant in the interval $(0,1)$ and random variables $V_1, V_2, \ldots$ are assumed to be independent and identically distributed, with each V_k also independent of X.

Problem 1: It can be shown (you need not do so) that, for any *specified* integer $m \geq 1$,

$$Y_m = \alpha^m X + \sum_{i=1}^{m} \alpha^{m-i} V_i \quad .$$

For the following questions, answers with sums or products involving constants and other known parameters are acceptable.

(a) (14 points) Supposing $\mathbf{E}[V_1] = \mu_V$, express $\mathbf{E}[Y_m]$ in terms of parameters α, m and μ_V.

(b) (14 points) Denote by $M_V(s)$ a *known* transform associated with random variable V_1. Express $\mathbf{E}[e^{sY_m}]$ in terms of parameters σ_X, α, m and $M_V(s)$.

(c) (14 points) Let N be a discrete random variable, known to be independent of all the V_k's as well as X, with a PMF that is given by

$$p_N(n) = \begin{cases} \frac{3}{4} & , \quad n = m \\ \frac{1}{4} & , \quad n = m+1 \\ 0 & , \quad \text{otherwise} \end{cases} \qquad .$$

Express, in terms of parameters σ_X, α, m and $M_V(s)$, the transform of the PDF that describes random variable Y_N.

Problem 2: It is known that $\mathbf{E}[V_1] = 0$ and that $\text{var}(Y_0) = \text{var}(Y_k)$ for all $k \geq 1$.

(a) (14 points) Express $\text{var}(V_1)$ in terms of parameters σ_X and α.

(b) (14 points) Provide a PDF, $f_V(v)$, for the identically distributed V_k's such that all of the Y_k's are Gaussian or explain why you cannot provide any such PDF.

(c) (14 points) For $i \geq 2$, express $\mathbf{E}[Y_i Y_{i-1} \mid Y_{i-2}]$ in terms of parameters σ_X and α.

(d) (14 points) For all $k \geq 1$, let $\hat{Y}_{k-1} = g(Y_k)$ denote the optimal least squares estimator of Y_{k-1} based on Y_k. Let $g_L(Y_k)$ denote the *linear* least squares approximation to $\hat{Y}_{k-1}$. Express $g_L(Y_k)$ in terms of parameters σ_X and α.