Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2002)

Quiz 2 Announcements

Quiz 2: (closed-book; two *handwritten* 2-sided 8.5x11 formula sheets and calculator permitted)
NOTE: Walker Gymnasium is unavailable; your quiz location depends on your TA!

Time: 50 minutes

Content: All topics, with emphasis on those since Quiz 1, discussed in
Lectures 1 through 16
Textbook chapters 1 through 4
Recitations 1 through 8
Tutorials 1 through 8
Problem Sets 1 through 7

Marathon Office Hours: The TAs will jointly hold office hours for 8 hours.
A schedule will be posted on the web soon.

Optional Quiz Review Session: There will be two identical, two-hour quiz review sessions administered by the TAs. The session will consist of two parts. In the first hour, a concise overview of the theory will be presented. In the second hour, a set of practice problems will be solved. The quiz review is completely optional, but it is usually a good idea to attend and reinforce your understanding of the material and perhaps gain some insight you did not have before. (Solving the attached practice quiz *before* you look at the solutions may also be a good idea, as this may indicate areas which need additional study.) Details for the quiz review sessions are:

Time: Two 2 hours (identical sessions)

The problems to be solved in the quiz review are attached, as is a practice quiz with solutions. We strongly recommend working the quiz review problems before coming to the quiz review.

Quiz 2 Review Problems

1. Random variables X and Y are independent and are described by the probability density functions $f_X(x)$ and $f_Y(y)$:

$$f_X(x) = \begin{cases} 1, & 0 < x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} 1, & 0 < y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Stations A and B are connected by two *parallel* message channels. A message from A to B is sent over both channels at the same time. Random variables X and Y represent the message delays in hours over parallel channels 1 and 2, respectively.

A message is considered "received" as soon as it arrives on any one channel, and it is considered "verified" as soon as it has arrived over both channels.

(a) Determine the probability that a message is received within 15 minutes after it is sent.

(b) Determine the probability that the message is received but not verified within 15 minutes after it is sent.

(c) Let T represent the time in hours between transmission at A and verification at B. Determine the CDF $F_T(t)$, and then differentiate it to obtain the PDF $f_T(t)$.

(d) If the attendant at B leaves for a 15-minute coffee break right after the message is received, what is the probability that he is present at the proper time for verification?

(e) The management wishes to have the maximum probability of having the attendant present for *both* reception and verification. Would they do better to let him take his coffee break as described above or simply allow him to go home 45 minutes after transmission?

2. The wombat club has N members, where N is a random variable with PMF

$$p_N(n) = p^{n-1}(1-p) \quad \text{for } n = 1, 2, 3, \ldots.$$

On the second Tuesday night of every month, the club holds a meeting. Each wombat member attends the meeting with probability q, independently of all the other members. If a wombat attends the meeting, then it brings an amount of money, M, which is a continuous random variable with PDF

$$f_M(m) = \lambda e^{-\lambda m} \quad \text{for } m \geq 0.$$

N, M, and whether each wombat member attends are all independent. Determine:

(a) The expectation and variance of the number of wombats showing up to the meeting.

(b) The probability that only one wombat shows up to the meeting.

(c) The transform of the PDF for the total amount of money brought to the meeting.

3. Oscar's dog has, yet again, run away from him. But, this time, Oscar will be using modern technology to aid him in his search. Oscar knows that X, the distance between him and his dog, is a random variable with a uniform distribution between five and ten miles. Oscar decides to use his pocket GPS device to help him pinpoint the dog's location.

Unfortunately, Oscar bought a cheap device and the measurement that Oscar receives is noisy, where the additive noise W, assumed to be independent of the actual location of the dog, is modeled as uniformly distributed between negative one and one. The measurement that Oscar reads on his display is random variable

$$Y = X + W \quad .$$

(a) Determine an estimator $g(Y)$ of X that minimizes $E[(X - g(Y))^2]$ for all possible measurement values y. Provide a plot of this optimal estimator as a function of y.

(b) Determine the *linear* least squares estimator of X based on Y. Plot this estimator and compare it with the estimator from part (a). [For comparison, just plot the two estimators on the same graph and make some comments].

PRACTICE QUIZ 2 (FROM SPRING 2002)

Problem 1: (2 points)
Write your TA's name on the front of the booklet.
Note: Some useful integrals:

$$\int_0^\infty xe^{-\lambda x}\,dx = \frac{1}{\lambda^2}, \qquad \int_0^\infty x^2e^{-\lambda x}\,dx = \frac{2}{\lambda^3}$$

Problem 2:
The random variable X is exponential with parameter 1. Given the value x of X, the random variable Y is exponential with parameter equal to x.

(a) (10 points) Find the joint PDF of X and Y.

(b) (10 points) Find the marginal PDF of Y.

(c) (11 points) Find the conditional PDF of X, given that $Y = 2$.

(d) (10 points) What is the least squares estimate of X, given that $Y = 2$?

(e) (10 points) Find the conditional PDF of Y, given that $X = 3$ and that $Y \geq 2$,

(f) (11 points) Find the PDF of e^{3X} (Show your work).

(g) (12 points) Let X_1, X_2, and X_3 be three independent random variables, each with the same distribution as X. Find the transform of $2(X_1 + X_2 + X_3) + 5$.

(h) **Do this part, part (h), ONLY if you are taking the graduate version, 6.431.** (12 points) Let the random variable Z be $Z = XY$. Find the transform of Z.

Problem 3:
The random variables N, $X_1, X_2, \ldots$ are all independent. The random variable N has a geometric distribution with parameter $p = 1/3$, so that

$$\mathbf{E}[N] = 3, \qquad \mathrm{var}(N) = 6.$$

The random variables X are identically distributed with

$$\mathbf{E}[X_i] = 4, \qquad \mathrm{var}(X_i) = 1.$$

(a) (12 points) Find the conditional mean and variance of $X_1 + \cdots + X_N$ given that $N \geq 5$.

(b) (12 points) Find the mean of $N(X_1 + \cdots + X_N)$.

Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2002)

Practice Quiz 2 Solutions (from Spring 2002)

Problem 1: (2 pts) Answers may vary.

Problem 2:

(a) (10 pts) $f_{X,Y}(x,y) = f_{Y|X}(y|x)f_X(x) = xe^{-xy}e^{-x} = \begin{cases} xe^{-(y+1)x} & x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

(b) (10 pts) $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_0^{\infty} xe^{-(y+1)x}dx = \begin{cases} \frac{1}{(y+1)^2} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

(c) (11 pts) $f_{X|Y}(x|2) = \frac{f_{X,Y}(x,2)}{f_Y(2)} = \frac{xe^{-(2+1)x}}{\frac{1}{(2+1)^2}} = \begin{cases} 9xe^{-3x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

(d) (10 pts) The least squares estimate of X given $Y = 2$ is $\mathbf{E}[X|Y = 2]$.

$\mathbf{E}[X|Y = 2] = \int_{-\infty}^{\infty} xf_{X|Y}(x|2)dx = \int_0^{\infty} 9x^2e^{-3x}dx = 9\frac{2}{27} = \frac{2}{3}$

(e) (10 pts) Given $X = 3$, Y is exponential with parameter 3. If in addition we are given $Y \geq 2$, then by the memoryless property Y is an exponential shifted by 2.

$f_{Y|X=3,Y\geq 2}(y) = \begin{cases} 3e^{-3(y-2)} & y \geq 2 \\ 0 & \text{otherwise} \end{cases}$

(f) (11 pts) $F_{e^{3X}}(w) = P(e^{3X} \leq w) = P(X \leq \ln w^{\frac{1}{3}}) = 1 - e^{-\ln w^{\frac{1}{3}}} = 1 - w^{-\frac{1}{3}}$

$f_{e^{3X}}(w) = \frac{d}{dw}F_{e^{3X}}(w) = \begin{cases} \frac{1}{3}w^{-\frac{4}{3}} & w \geq 1 \\ 0 & \text{otherwise} \end{cases}$

(g) (12 pts) $M_{2(X_1+X_2+X_3)+5}(s) = e^{5s}M_{X_1+X_2+X_3}(2s) = e^{5s}(M_X(2s))^3 = e^{5s}(\frac{1}{1-2s})^3$

(h) 6.431 ONLY (12 pts) $M_{XY}(s) = \mathbf{E}[e^{sXY}] = \int_{-\infty}^{\infty} \mathbf{E}[e^{sXY}|X = x]f_X(x)dx = \int_0^{\infty} M_{Y|X=x}(sx)e^{-x}dx = \int_0^{\infty} \frac{x}{x-sx}e^{-x}dx = \frac{1}{1-s}$

Problem 3:

(a) (12 pts) Because N and X_i are independent, X_i's distribution is unaffected by the event $N \geq 5$, meaning $\mathbf{E}[X|N \geq 5] = \mathbf{E}[X] = 4$ and $\text{var}(X|N \geq 5) = \text{var}(X) = 1$. By the memoryless property, N is a geometric shifted by 4 when given $N \geq 5$, meaning $\mathbf{E}[N|N \geq 5] = \mathbf{E}[N + 4] = \mathbf{E}[N] + 4 = 7$ and $\text{var}(N|N \geq 5) = \text{var}(N + 4) = \text{var}(N) = 6$.

$\mathbf{E}[X_1 + \ldots + X_N|N \geq 5] = \mathbf{E}[X|N \geq 5]\mathbf{E}[N|N \geq 5] = 4 \cdot 7 = 28$

$\text{var}(X_1 + \ldots + X_N|N \geq 5) = \mathbf{E}[N|N \geq 5]\text{var}(X|N \geq 5) + \mathbf{E}[X|N \geq 5]^2\text{var}(N|N \geq 5) = 7 \cdot 1 + 4^2 \cdot 6 = 103$

(b) (12 pts) $\mathbf{E}[N(X_1 + \ldots + X_N)] = \mathbf{E}[\mathbf{E}[N(X_1 + \ldots + X_N)|N]] = \mathbf{E}[N^2\mathbf{E}[X]] = \mathbf{E}[N^2]\mathbf{E}[X] = (\text{var}(N) + \mathbf{E}[N]^2)\mathbf{E}[X] = (6 + 3^2)4 = 60$