Probability and Stochastic Processes
Homework Chapter 12 Solutions

Problem Solutions : Yates and Goodman, 12.1.1 12.1.4 12.3.2 12.4.3 12.5.3 12.5.6 12.6.1 12.9.1 12.9.4 12.10.1 12.10.6 12.11.1 12.11.3 12.11.5 and 12.11.9

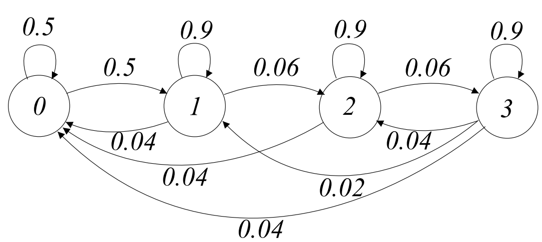
Problem 12.1.1 Solution

From the given Markov chain, the state transition matrix is

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0.25 & 0.25 & 0.5 \end{bmatrix} \tag{1}$$

Problem 12.1.4 Solution

Based on the problem statement, the state of the wireless LAN is given by the following Markov chain:



The Markov chain has state transition matrix

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.04 & 0.9 & 0.06 & 0 \\ 0.04 & 0 & 0.9 & 0.06 \\ 0.04 & 0.02 & 0.04 & 0.9 \end{bmatrix}. \tag{1}$$

Problem 12.3.2 Solution

At time $n-1$, let $p_i(n-1)$ denote the state probabilities. By Theorem 12.4, the probability of state k at time n is

$$p_k(n) = \sum_{i=0}^{\infty} p_i(n-1) P_{ik} \tag{1}$$

Since $P_{ik} = q$ for every state i,

$$p_k(n) = q \sum_{i=0}^{\infty} p_i(n-1) = q \tag{2}$$

Thus for any time $n > 0$, the probability of state k is q.

Problem 12.4.3 Solution

The idea behind this claim is that if states j and i communicate, then sometimes when we go from state j back to state j, we will pass through state i. If $E[T_{ij}] = \infty$, then on those occasions we pass through i, the expected time to go to back to j will be infinite. This would suggest $E[T_{jj}] = \infty$ and thus state j would not be positive recurrent. Using a math to prove this requires a little bit of care.

Suppose $E[T_{ij}] = \infty$. Since i and j communicate, we can find n, the smallest nonnegative integer such that $P_{ji}(n) > 0$. Given we start in state j, let G_i denote the event that we go through state i on our way back to j. By conditioning on G_j,

$$E\left[T_{jj}\right] = E\left[T_{jj}|G_i\right] P\left[G_i\right] + E\left[T_{jj}|G_i^c\right] P\left[G_i^c\right] \tag{1}$$

Since $E[T_{jj}|G_i^c]P[G_i^c] \geq 0$,

$$E\left[T_{jj}\right] \geq E\left[T_{jj}|G_i\right] P\left[G_i\right] \tag{2}$$

Given the event G_i, $T_{jj} = T_{ji} + T_{ij}$. This implies

$$E\left[T_{jj}|G_i\right] = E\left[T_{ji}|G_i\right] + E\left[T_{ij}|G_i\right] \geq E\left[T_{ij}|G_i\right] \tag{3}$$

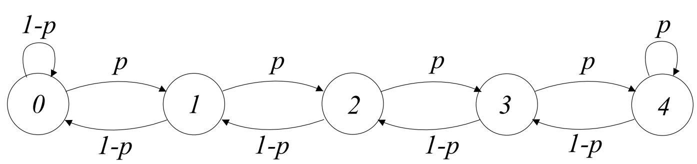
Since the random variable T_{ij} assumes that we start in state i, $E[T_{ij}|G_i] = E[T_{ij}]$. Thus $E[T_{jj}|G_i] \geq E[T_{ij}]$. In addition, $P[G_i] \geq P_{ji}(n)$ since there may be paths with more than n hops that take the system from state j to i. These facts imply

$$E\left[T_{jj}\right] \geq E\left[T_{jj}|G_i\right] P\left[G_i\right] \geq E\left[T_{ij}\right] P_{ji}(n) = \infty \tag{4}$$

Thus, state j is not positive recurrent, which is a contradiction. Hence, it must be that $E[T_{ij}] < \infty$.

Problem 12.5.3 Solution

From the problem statement, the Markov chain is



The self-transitions in state 0 and state 4 guarantee that the Markov chain is aperiodic. Since the chain is also irreducible, we can find the stationary probabilities by solving $\boldsymbol{\pi} = \boldsymbol{\pi}'\mathbf{P}$; however, in this problem it is simpler to apply Theorem 12.13. In particular, by partitioning the chain between states i and $i+1$, we obtain

$$\pi_i p = \pi_{i+1}(1-p). \tag{1}$$

This implies $\pi_{i+1} = \alpha\pi_i$ where $\alpha = p/(1-p)$. It follows that $\pi_i = \alpha^i \pi_0$. REquiring the stationary probabilities to sum to 1 yields

$$\sum_{i=0}^{4} \pi_i = \pi_0(1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4) = 1. \tag{2}$$

This implies

$$\pi_0 = \frac{1-\alpha^5}{1-\alpha} \tag{3}$$

Thus, for $i = 0, 1, \ldots, 4$,

$$\pi_i = \frac{1-\alpha^5}{1-\alpha}\alpha^i = \frac{1-\left(\frac{p}{1-p}\right)^5}{1-\left(\frac{p}{1-p}\right)}\left(\frac{p}{1-p}\right)^i. \tag{4}$$

Problem 12.5.6 Solution

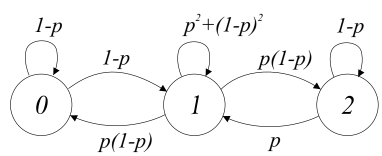
This system has three states:

0 front teller busy, rear teller idle

1 front teller busy, rear teller busy

2 front teller idle, rear teller busy

We will assume the units of time are seconds. Thus, if a teller is busy one second, the teller will become idle in th next second with probability $p = 1/120$. The Markov chain for this system is



We can solve this chain very easily for the stationary probability vector $\boldsymbol{\pi}$. In particular,

$$\pi_0 = (1-p)\pi_0 + p(1-p)\pi_1 \tag{1}$$

This implies that $\pi_0 = (1-p)\pi_1$. Similarly,

$$\pi_2 = (1-p)\pi_2 + p(1-p)\pi_1 \tag{2}$$

yields $\pi_2 = (1-p)\pi_1$. Hence, by applying $\pi_0 + \pi_1 + \pi_2 = 1$, we obtain

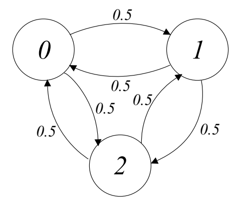
$$\pi_0 = \pi_2 = \frac{1-p}{3-2p} = 119/358 \tag{3}$$

$$\pi_1 = \frac{1}{3-2p} = 120/358 \tag{4}$$

The stationary probability that both tellers are busy is $\pi_1 = 120/358$.

Problem 12.6.1 Solution

Equivalently, we can prove that if $P_{ii} \neq 0$ for some i, then the chain cannot be periodic. So, suppose for state i, $P_{ii} > 0$. Since $P_{ii} = P_{ii}(1)$, we see that the largest d that divides n for all n such that $P_{ii}(n) > 0$ is $d = 1$. Hence, state i is aperiodic and thus the chain is aperiodic.
The converse that $P_{ii} = 0$ for all i implies the chain is periodic is false. As a counterexample, consider the simple chain on the right with $P_{ii} = 0$ for each i. Note that $P_{00}(2) > 0$ and $P_{00}(3) > 0$. The largest d that divides both 2 and 3 is $d = 1$. Hence, state 0 is aperiodic. Since the chain has one communicating class, the chain is also aperiodic.

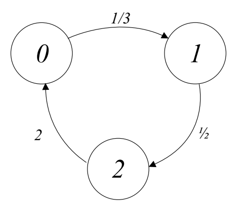


Problem 12.9.1 Solution

From the problem statement, we learn that in each state i, the tiger spends an exponential time with parameter λ_i. When we measure time in hours,

$$\lambda_0 = q_{01} = 1/3 \qquad \lambda_1 = q_{12} = 1/2 \qquad \lambda_2 = q_{20} = 2 \tag{1}$$

The corresponding continous time Markov chain is shown below:



The state probabilities satisfy

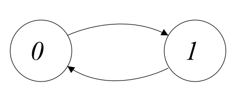
$$\frac{1}{3}p_0 = 2p_2 \qquad \frac{1}{2}p_1 = \frac{1}{3}p_0 \qquad p_0 + p_1 + p_2 = 1 \tag{2}$$

The solution is

$$\begin{bmatrix} p_0 & p_1 & p_2 \end{bmatrix} = \begin{bmatrix} 6/11 & 4/11 & 1/11 \end{bmatrix} \tag{3}$$

Problem 12.9.4 Solution

In this problem, we build a two-state Markov chain such that the system in state $i \in \{0, 1\}$ if the most recent arrival of either Poisson process is type i. Note that if the system is in state 0, transitions to state 1 occur with rate λ_1. If the system is in state 1, transitions to state 0 occur at rate λ_0. The continuous time Markov chain is just



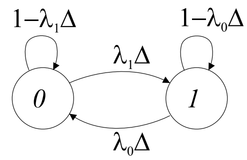
The stationary probabilities satisfy $p_0\lambda_1 = p_1\lambda_0$. Thus $p_1 = (\lambda_1/\lambda_0)p_0$. Since $p_0 + p_1 = 1$, we have that

$$p_0 + (\lambda_1/\lambda_0)p_0 = 1. \tag{1}$$

This implies

$$p_0 = \frac{\lambda_0}{\lambda_0 + \lambda_1}, \qquad p_1 = \frac{\lambda_1}{\lambda_0 + \lambda_1}. \tag{2}$$

It is also possible to solve this problem using a discrete time Markov chain. One way to do this is to assume a very small time step Δ. In state 0, a transition to state 1 occurs with probability $\lambda_1\Delta$; otherwise the system stays in state 0 with probability $1-\lambda_1\Delta$. Similarly, in state 1, a transition to state 0 occurs with probability $\lambda_0\Delta$; otherwise the system stays in state 1 with probability $1-\lambda_0\Delta$. Here is the Markov chain for this discrete time system:



Not surprisingly, the stationary probabilities for this discrete time system are

$$\pi_0 = \frac{\lambda_0}{\lambda_0 + \lambda_1}, \qquad \pi_1 = \frac{\lambda_1}{\lambda_0 + \lambda_1}. \tag{3}$$

Problem 12.10.1 Solution

In Equation (12.93), we found that the blocking probability of the $M/M/c/c$ queue was given by the Erlang-B formula

$$P[B] = P_N(c) = \frac{\rho^c/c!}{\sum_{k=0}^{c} \rho^k/k!} \tag{1}$$

The parameter $\rho = \lambda/\mu$ is the normalized load. When $c = 2$, the blocking probability is

$$P[B] = \frac{\rho^2/2}{1+\rho+\rho^2/2} \tag{2}$$

Setting $P[B] = 0.1$ yields the quadratic equation

$$\rho^2 - \frac{2}{9}\rho - \frac{2}{9} = 0 \tag{3}$$

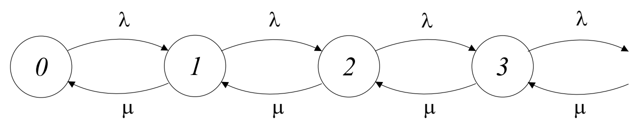
The solutions to this quadratic are

$$\rho = \frac{1 \pm \sqrt{19}}{9} \tag{4}$$

The meaningful nonnegative solution is $\rho = (1+\sqrt{19})/9 = 0.5954$.

Problem 12.10.6 Solution

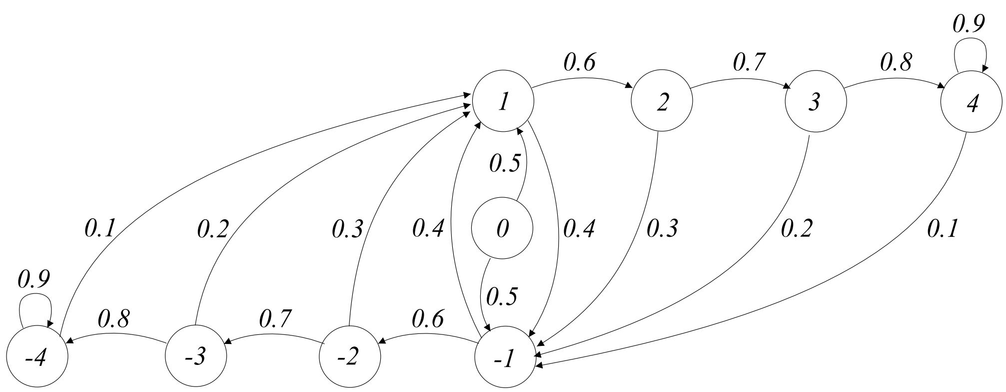
The LCFS queue operates in a way that is quite different from the usual first come, first served queue. However, under the assumptions of exponential service times and Poisson arrivals, customers arrive at rate λ and depart at rate μ, no matter which service discipline is used. The Markov chain for the LCFS queue is the same as the Markov chain for the $M/M/1$ first come, first served queue:



It would seem that the LCFS queue should be less efficient than the ordinary $M/M/1$ queue because a new arrival causes us to discard the work done on the customer in service. This is not the case, however, because the memoryless property of the exponential PDF implies that no matter how much service had already been performed, the remaining service time remains identical to that of a new customer.

Problem 12.11.1 Solution

Here is the Markov chain describing the free throws.



Note that state 4 corresponds to "4 or more consecutive successes" while state -4 corresponds to "4 or more consecutive misses." We denote the stationary probabilities by the vector

$$\boldsymbol{\pi} = \begin{bmatrix}\pi_{-4} & \pi_{-3} & \pi_{-2} & \pi_{-1} & \pi_0 & \pi_1 & \pi_2 & \pi_3 & \pi_4\end{bmatrix}'. \tag{1}$$

For this vector $\boldsymbol{\pi}$, the state transition matrix is

$$\mathbf{P} = \begin{bmatrix}
0.9 & 0 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0 \\
0.8 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 \\
0 & 0.7 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0 \\
0 & 0 & 0.6 & 0 & 0 & 0.4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.4 & 0 & 0 & 0.6 & 0 & 0 \\
0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0.7 & 0 \\
0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0.8 \\
0 & 0 & 0 & 0.1 & 0 & 0 & 0 & 0 & 0.9
\end{bmatrix}. \tag{2}$$

To solve the problem at hand, we divide the work into two functions; `freethrowmat(n)` returns the `n` step transition matrix and `freethrowp(n)` that calculates the probability of a success on the free throw `n`.

```
function Pn=freethrowmat(n);
P=[0.9 0   0   0   0  0.1 0   0   0;...
   0.8 0   0   0   0  0.2 0   0   0;...
   0   0.7 0   0   0  0.3 0   0   0;...
   0   0   0.6 0   0  0.4 0   0   0;...
   0   0   0   0.5 0  0.5 0   0   0;...
   0   0   0   0.4 0   0  0.6 0   0;...
   0   0   0   0.3 0   0  0   0.7 0;...
   0   0   0   0.2 0   0  0   0   0.8;...
   0   0   0   0.1 0   0  0   0   0.9];
Pn=P^n;
```

```
function ps=freethrowp(n);
PP=freethrowmat(n-1);
p0=[zeros(1,4) 1 ...
        zeros(1,4)];
ps=p0*PP*0.1*(1:9)';
```

In `freethrowp.m`, `p0` is the initial state probability row vector $\boldsymbol{\pi}'(0)$. Thus `p0*PP` is the state probability row vector after $n-1$ free throws. Finally, `p0*PP*0.1*(1:9)'` multiplies the state probability vector by the conditional probability of successful free throw given the current state. The answer to our problem is simply

```
>> freethrowp(11)
ans =
    0.5000
>>
```

In retrospect, the calculations are unnecessary! Because the system starts in state 0, symmetry of the Markov chain dictates that states $-k$ and k will have the same probability at every time step. Because state $-k$ has success probability $0.5 - 0.1k$ while state k has success probability $0.5 + 0.1k$, the conditional success probability given the system is in state $-k$ or k is 0.5. Averaged over $k = 1, 2, 3, 4$, the average success probability is still 0.5.

Comment: Perhaps finding the stationary distribution is more interesting. This is done fairly easily:

```
>> p=dmcstatprob(freethrowmat(1));
>> p'
ans =
  0.3123 0.0390 0.0558 0.0929 0 0.0929 0.0558 0.0390 0.3123
```

About sixty precent of the time the shooter has either made four or more consecutive free throws or missed four or more free throws. On the other hand, one can argue that in a basketball game, a shooter rarely gets to take more than a half dozen (or so) free throws, so perhaps the stationary distribution isn't all that interesting.

Problem 12.11.3 Solution

Although the inventory system in this problem is relatively simple, the performance analysis is suprisingly complicated. We can model the system as a Markov chain with state X_n equal to the number of brake pads in stock at the start of day n. Note that the range of X_n is $S_X = \{50, 51, \ldots, 109\}$. To evaluate the system, we need to find the state transition matrix for the Markov chain. We express the transition probabilities in terms of $P_K(\cdot)$, the PMF of the number of brake pads ordered on an arbitary day. In state i, there are two possibilities:

- If $50 \leq i \leq 59$, then there will be $\min(i, K)$ brake pads sold. At the end of the day, the number of pads remaining is less than 60, and so 50 more pads are delivered overnight.

Thus the next state is $j = 50$ if $K \geq i$ pads are ordered, i pads are sold and 50 pads are delivered overnight. On the other hand, if there are $K < i$ orders, then the next state is $j = i - K + 50$. In this case,

$$P_{ij} = \begin{cases} P[K \geq i] & j = 50, \\ P_K(50 + i - j) & j = 51, 52, \ldots, 50 + i. \end{cases} \tag{1}$$

- If $60 \leq i \leq 109$, then there are several subcases:
  - $j = 50$: If there are $K \geq i$ orders, then all i pads are sold, 50 pads are delivered overnight, and the next state is $j = 50$. Thus

    $$P_{ij} = P[K \geq i], \qquad j = 50. \tag{2}$$

  - $51 \leq j \leq 59$: If $50 + i - j$ pads are sold, then $j - 50$ pads ar left at the end of the day. In this case, 50 pads are delivered overnight, and the next state is j with probability

    $$P_{ij} = P_K(50 + i - j), \qquad j = 51, 52, \ldots, 59. \tag{3}$$

  - $60 \leq j \leq i$: If there are $K = i - j$ pads ordered, then there will be $j \geq 60$ pads at the end of the day and the next state is j. On the other hand, if $K = 50 + i - j$ pads are ordered, then there will be $i - (50 + i - j) = j - 50$ pads at the end of the day. Since $60 \leq j \leq 109$, $10 \leq j - 50 \leq 59$, there will be 50 pads delivered overnight and the next state will be j. Thus

    $$P_{ij} = P_K(i - j) + P_K(50 + i - j), \qquad j = 60, 61, \ldots, i. \tag{4}$$

  - For $i < j \leq 109$, state j can be reached from state i if there $50 + i - j$ orders, leaving $i - (50 + i - j) = j - 50$ in stock at the end of the day. This implies 50 pads are delivered overnight and the next stage is j. the probability of this event is

    $$P_{ij} = P_K(50 + i - j), \qquad j = i + 1, i + 2, \ldots, 109. \tag{5}$$

We can summarize these observations in this set of state transition probabilities:

$$P_{ij} = \begin{cases} P[K \geq i] & 50 \leq i \leq 109, j = 50, \\ P_K(50 + i - j) & 50 \leq i \leq 59, 51 \leq j \leq 50 + i, \\ P_K(50 + i - j) & 60 \leq i \leq 109, 51 \leq j \leq 59, \\ P_K(i - j) + P_K(50 + i - j) & 60 \leq i \leq 109, 60 \leq j \leq i, \\ P_K(50 + i - j) & 60 \leq i \leq 108, i + 1 \leq j \leq 109 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Note that the "0 otherwise" rule comes into effect when $50 \leq i \leq 59$ and $j > 50 + i$. To simplify these rules, we observe that $P_K(k) = 0$ for $k < 0$. This implies $P_K(50 + i - j) = 0$

for $j > 50 + i$. In addition, for $j > i$, $P_K(i - j) = 0$. These facts imply that we can write the state transition probabilities in the simpler form:

$$P_{ij} = \begin{cases} P\left[K \geq i\right] & 50 \leq i \leq 109, j = 50, \\ P_K\left(50 + i - j\right) & 50 \leq i \leq 59, 51 \leq j \\ P_K\left(50 + i - j\right) & 60 \leq i \leq 109, 51 \leq j \leq 59, \\ P_K\left(i - j\right) + P_K\left(50 + i - j\right) & 60 \leq i \leq 109, 60 \leq j \end{cases} \tag{7}$$

Finally, we make the definitions

$$\beta_i = P\left[K \geq i\right], \qquad \gamma_k = P_K\left(50 + k\right), \qquad \delta_k = P_K\left(k\right) + P_K\left(50 + k\right). \tag{8}$$

With these definitions, the state transition probabilities are

$$P_{ij} = \begin{cases} \beta_i & 50 \leq i \leq 109, j = 50, \\ \gamma_{i-j} & 50 \leq i \leq 59, 51 \leq j \\ \gamma_{i-j} & 60 \leq i \leq 109, 51 \leq j \leq 59, \\ \delta_{i-j} & 60 \leq i \leq 109, 60 \leq j \end{cases} \tag{9}$$

Expressed as a table, the state transition matrix $\mathbf{P}$ is

$$\begin{array}{c|cccccccccc}
i \backslash j & 50 & 51 & \cdots & 59 & 60 & \cdots & \cdots & \cdots & \cdots & 109 \\ \hline
50 & \beta_{50} & \gamma_{-1} & \cdots & \gamma_{-9} & \cdots & \cdots & \cdots & \cdots & \cdots & \gamma_{-59} \\
51 & \beta_{51} & \gamma_0 & \ddots & \vdots & \ddots & & & & & \vdots \\
\vdots & \vdots & \vdots & \ddots & \gamma_{-1} & & \ddots & & & \ddots & \vdots \\
59 & \beta_{59} & \gamma_8 & \cdots & \gamma_0 & \gamma_{-1} & \cdots & \gamma_{-9} & \cdots & \cdots & \gamma_{-50} \\
60 & \beta_{60} & \gamma_9 & \cdots & \gamma_1 & \delta_0 & \cdots & & \delta_{-9} & \cdots & \delta_{-49} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & & & \ddots & \vdots \\
\vdots & \vdots & \vdots & & \gamma_9 & \vdots & & & & & \delta_{-9} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \delta_9 & & & & & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & & & \ddots & \vdots \\
109 & \beta_{109} & \gamma_{58} & \cdots & \gamma_{50} & \delta_{49} & \cdots & \delta_9 & \cdots & \cdots & \delta_0
\end{array} \tag{10}$$

In terms of MATLAB, all we need to do is to encode the matrix $\mathbf{P}$, calculate the stationary probability vector $\boldsymbol{\pi}$, and then calculate $E[Y]$, the expected number of pads sold on a typical day. To calculate $E[Y]$, we use iterated expectation. The number of pads ordered is the Poisson random variable K. We assume that on a day n that $X_n = i$ and we calculate the conditional expectation

$$E\left[Y|X_n = i\right] = E\left[\min(K, i)\right] = \sum_{j=0}^{i-1} jP_K\left(j\right) + iP\left[K \geq i\right]. \tag{11}$$

Since only states $i \geq 50$ are possible, we can write

$$E\left[Y|X_n = i\right] = \sum_{j=0}^{48} jP_K\left(j\right) + \sum_{j=49}^{i-1} jP_K\left(j\right) + iP\left[K \geq i\right]. \tag{12}$$

Finally, we assume that on a typical day n, the state of the system X_n is described by the stationary probabilities $P[X_n = i] = \pi_i$ and we calculate

$$E[Y] = \sum_{i=50}^{109} E[Y|X_n = i]\,\pi_i. \tag{13}$$

These calculations are given in this MATLAB program:

```
function [pstat,ey]=brakepads(alpha);
s=(50:109)';
beta=1-poissoncdf(alpha,s-1);
grow=poissonpmf(alpha,50+(-1:-1:-59));
gcol=poissonpmf(alpha,50+(-1:58));
drow=poissonpmf(alpha,0:-1:-49);
dcol=poissonpmf(alpha,0:49);
P=[beta,toeplitz(gcol,grow)];
P(11:60,11:60)=P(11:60,11:60)...
    +toeplitz(dcol,drow);
pstat=dmcstatprob(P);
[I,J]=ndgrid(49:108,49:108);
G=J.*(I>=J);
EYX=(G*gcol)+(s.*beta);
pk=poissonpmf(alpha,0:48);
EYX=EYX+(0:48)*pk;
ey=(EYX')*pstat;
```

The first half of `brakepads.m` constructs $\mathbf{P}$ to calculate the stationary probabilities. The first column of $\mathbf{P}$ is just the vector

$$\texttt{beta} = \begin{bmatrix}\beta_{50} & \cdots & \beta_{109}\end{bmatrix}'. \tag{14}$$

The rest of $\mathbf{P}$ is easy to construct using `toeplitz` function. We first build an asymmetric Toeplitz matrix with first row and first column

$$\texttt{grow} = \begin{bmatrix}\gamma_{-1} & \gamma_{-2} & \cdots & \gamma_{-59}\end{bmatrix} \tag{15}$$

$$\texttt{gcol} = \begin{bmatrix}\gamma_{-1} & \gamma_{0} & \cdots & \gamma_{58}\end{bmatrix}' \tag{16}$$

Note that $\delta_k = P_K(k) + \gamma_k$. Thus, to construct the Toeplitz matrix in the lower right corner of $\mathbf{P}$, we simply add the Toeplitz matrix corresponding to the missing $P_K(k)$ term. The second half of `brakepads.m` calculates $E[Y]$ using the iterated expectation. Note that

$$\texttt{EYX} = \begin{bmatrix}E[Y|X_n = 50] & \cdots & E[Y|X_n = 109]\end{bmatrix}'. \tag{17}$$

The somewhat convoluted code becomes clearer by noting the following correspondences:

$$E[Y|X_n = i] = \underbrace{\sum_{j=0}^{48} jP_K(j)}_{\texttt{(0:48)*pk}} + \underbrace{\sum_{j=49}^{i-1} jP_K(j)}_{\texttt{G*gcol}} + \underbrace{iP[K \geq i]}_{\texttt{s.*beta}}. \tag{18}$$

To find $E[Y]$, we execute the commands:

```
>> [ps,ey]=brakepads(50);
>> ey
ey =
   49.4154
>>
```

We see that although the store receives 50 orders for brake pads on average, the average number sold is only 49.42 because once in awhile the pads are out of stock. Some experimentation will show that if the expected number of orders α is significantly less than 50, then the expected number of brake pads sold each days is very close to α. On the other hand, if $\alpha \gg 50$, then the each day the store will run out of pads and will get a delivery of 50 pads ech night. The expected number of unfulfilled orders will be very close to $\alpha - 50$.

Note that a new inventory policy in which the overnight delivery is more than 50 pads or the threshold for getting a shipment is more than 60 will reduce the expected numer of unfulfilled orders. Whether such a change in policy is a good idea depends on factors such as the carrying cost of inventory that are absent from our simple model.

Problem 12.11.5 Solution

Under construction.

Problem 12.11.9 Solution

Under construction.