ECE 541
Stochastic Signals and Systems
Problem Set 1 Solutions
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Problem Solutions : Yates and Goodman, 2.2.8 2.3.13 2.5.9 2.6.6 2.7.7 2.7.8 2.8.8 2.10.4 3.2.5 3.4.14 3.5.10 3.7.2 3.8.8 and 3.8.9

Problem 2.2.8 Solution

From the problem statement, a single is twice as likely as a double, which is twice as likely as a triple, which is twice as likely as a home-run. If p is the probability of a home run, then

$$P_B(4) = p \qquad P_B(3) = 2p \qquad P_B(2) = 4p \qquad P_B(1) = 8p \tag{1}$$

Since a hit of any kind occurs with probability of .300, $p + 2p + 4p + 8p = 0.300$ which implies $p = 0.02$. Hence, the PMF of B is

$$P_B(b) = \begin{cases} 0.70 & b = 0 \\ 0.16 & b = 1 \\ 0.08 & b = 2 \\ 0.04 & b = 3 \\ 0.02 & b = 4 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Problem 2.3.13 Solution

(a) Let S_n denote the event that the Sixers win the series in n games. Similarly, C_n is the event that the Celtics in in n games. The Sixers win the series in 3 games if they win three straight, which occurs with probability

$$P[S_3] = (1/2)^3 = 1/8 \tag{1}$$

The Sixers win the series in 4 games if they win two out of the first three games and they win the fourth game so that

$$P[S_4] = \binom{3}{2}(1/2)^3(1/2) = 3/16 \tag{2}$$

The Sixers win the series in five games if they win two out of the first four games and then win game five. Hence,

$$P[S_5] = \binom{4}{2}(1/2)^4(1/2) = 3/16 \tag{3}$$

By symmetry, $P[C_n] = P[S_n]$. Further we observe that the series last n games if either the Sixers or the Celtics win the series in n games. Thus,

$$P[N = n] = P[S_n] + P[C_n] = 2P[S_n] \tag{4}$$

Consequently, the total number of games, N, played in a best of 5 series between the Celtics and the Sixers can be described by the PMF

$$P_N(n) = \begin{cases} 2(1/2)^3 = 1/4 & n = 3 \\ 2\binom{3}{1}(1/2)^4 = 3/8 & n = 4 \\ 2\binom{4}{2}(1/2)^5 = 3/8 & n = 5 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

(b) For the total number of Celtic wins W, we note that if the Celtics get $w < 3$ wins, then the Sixers won the series in $3 + w$ games. Also, the Celtics win 3 games if they win the series in 3,4, or 5 games. Mathematically,

$$P[W = w] = \begin{cases} P[S_{3+w}] & w = 0, 1, 2 \\ P[C_3] + P[C_4] + P[C_5] & w = 3 \end{cases} \tag{6}$$

Thus, the number of wins by the Celtics, W, has the PMF shown below.

$$P_W(w) = \begin{cases} P[S_3] = 1/8 & w = 0 \\ P[S_4] = 3/16 & w = 1 \\ P[S_5] = 3/16 & w = 2 \\ 1/8 + 3/16 + 3/16 = 1/2 & w = 3 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

(c) The number of Celtic losses L equals the number of Sixers' wins W_S. This implies $P_L(l) = P_{W_S}(l)$. Since either team is equally likely to win any game, by symmetry, $P_{W_S}(w) = P_W(w)$. This implies $P_L(l) = P_{W_S}(l) = P_W(l)$. The complete expression of for the PMF of L is

$$P_L(l) = P_W(l) = \begin{cases} 1/8 & l = 0 \\ 3/16 & l = 1 \\ 3/16 & l = 2 \\ 1/2 & l = 3 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

Problem 2.5.9 Solution

In this "double-or-nothing" type game, there are only two possible payoffs. The first is zero dollars, which happens when we lose 6 straight bets, and the second payoff is 64 dollars which happens unless we lose 6 straight bets. So the PMF of Y is

$$P_Y(y) = \begin{cases} (1/2)^6 = 1/64 & y = 0 \\ 1 - (1/2)^6 = 63/64 & y = 64 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The expected amount you take home is

$$E[Y] = 0(1/64) + 64(63/64) = 63 \tag{2}$$

So, on the average, we can expect to break even, which is not a very exciting proposition.

Problem 2.6.6 Solution

The cellular calling plan charges a flat rate of \$20 per month up to and including the 30th minute, and an additional 50 cents for each minute over 30 minutes. Knowing that the time you spend on the phone is a geometric random variable M with mean $1/p = 30$, the PMF of M is

$$P_M(m) = \begin{cases} (1-p)^{m-1}p & m = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The monthly cost, C obeys

$$P_C(20) = P[M \leq 30] = \sum_{m=1}^{30} (1-p)^{m-1}p = 1 - (1-p)^{30} \tag{2}$$

When $M \geq 30$, $C = 20 + (M-30)/2$ or $M = 2C - 10$. Thus,

$$P_C(c) = P_M(2c - 10) \qquad c = 20.5, 21, 21.5, \ldots \tag{3}$$

The complete PMF of C is

$$P_C(c) = \begin{cases} 1 - (1-p)^{30} & c = 20 \\ (1-p)^{2c-10-1}p & c = 20.5, 21, 21.5, \ldots \end{cases} \tag{4}$$

Problem 2.7.7 Solution

We define random variable W such that $W = 1$ if the circuit works or $W = 0$ if the circuit is defective. (In the probability literature, W is called an indicator random variable.) Let R_s denote the profit on a circuit with standard devices. Let R_u denote the profit on a circuit with ultrareliable devices. We will compare $E[R_s]$ and $E[R_u]$ to decide which circuit implementation offers the highest expected profit.

The circuit with standard devices works with probability $(1-q)^{10}$ and generates revenue of k dollars if all of its 10 constituent devices work. We observe that we can we can express R_s as a function $r_s(W)$ and that we can find the PMF $P_W(w)$:

$$R_s = r_s(W) = \begin{cases} -10 & W = 0, \\ k - 10 & W = 1, \end{cases} \qquad P_W(w) = \begin{cases} 1 - (1-q)^{10} & w = 0, \\ (1-q)^{10} & w = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Thus we can express the expected profit as

$$\begin{aligned} E[r_s(W)] &= \sum_{w=0}^{1} P_W(w)\, r_s(w) && (2) \\ &= P_W(0)(-10) + P_W(1)(k-10) && (3) \\ &= (1-(1-q)^{10})(-10) + (1-q)^{10}(k-10) = (0.9)^{10}k - 10. && (4) \end{aligned}$$

For the ultra-reliable case,

$$R_u = r_u(W) = \begin{cases} -30 & W = 0, \\ k-30 & W = 1, \end{cases} \qquad P_W(w) = \begin{cases} 1-(1-q/2)^{10} & w = 0, \\ (1-q/2)^{10} & w = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Thus we can express the expected profit as

$$\begin{aligned} E\left[r_u(W)\right] &= \sum_{w=0}^{1} P_W(w)\, r_u(w) && (6)\\ &= P_W(0)(-30) + P_W(1)(k-30) && (7)\\ &= (1-(1-q/2)^{10})(-30) + (1-q/2)^{10}(k-30) = (0.95)^{10}k - 30 && (8) \end{aligned}$$

To determine which implementation generates the most profit, we solve $E[R_u] \geq E[R_s]$, yielding $k \geq 20/[(0.95)^{10} - (0.9)^{10}] = 80.21$. So for $k <$ \$80.21 using all standard devices results in greater revenue, while for $k >$ \$80.21 more revenue will be generated by implementing all ultra-reliable devices. That is, when the price commanded for a working circuit is sufficiently high, we should build more-expensive higher-reliability circuits.

If you have read ahead to Section 2.9 and learned about conditional expected values, you might prefer the following solution. If not, you might want to come back and review this alternate approach after reading Section 2.9.

Let W denote the event that a circuit works. The circuit works and generates revenue of k dollars if all of its 10 constituent devices work. For each implementation, standard or ultra-reliable, let R denote the profit on a device. We can express the expected profit as

$$E[R] = P[W]\,E[R|W] + P[W^c]\,E[R|W^c] \tag{9}$$

Let's first consider the case when only standard devices are used. In this case, a circuit works with probability $P[W] = (1-q)^{10}$. The profit made on a working device is $k-10$ dollars while a nonworking circuit has a profit of -10 dollars. That is, $E[R|W] = k-10$ and $E[R|W^c] = -10$. Of course, a negative profit is actually a loss. Using R_s to denote the profit using standard circuits, the expected profit is

$$E[R_s] = (1-q)^{10}(k-10) + (1-(1-q)^{10})(-10) = (0.9)^{10}k - 10 \tag{10}$$

And for the ultra-reliable case, the circuit works with probability $P[W] = (1-q/2)^{10}$. The profit per working circuit is $E[R|W] = k-30$ dollars while the profit for a nonworking circuit is $E[R|W^c] = -30$ dollars. The expected profit is

$$E[R_u] = (1-q/2)^{10}(k-30) + (1-(1-q/2)^{10})(-30) = (0.95)^{10}k - 30 \tag{11}$$

Not surprisingly, we get the same answers for $E[R_u]$ and $E[R_s]$ as in the first solution by performing essentially the same calculations. it should be apparent that indicator random variable W in the first solution indicates the occurrence of the conditioning event W in the second solution. That is, indicators are a way to track conditioning events.

Problem 2.7.8 Solution

(a) There are $\binom{46}{6}$ equally likely winning combinations so that

$$q = \frac{1}{\binom{46}{6}} = \frac{1}{9{,}366{,}819} \approx 1.07 \times 10^{-7} \tag{1}$$

(b) Assuming each ticket is chosen randomly, each of the $2n-1$ other tickets is independently a winner with probability q. The number of other winning tickets K_n has the binomial PMF

$$P_{K_n}(k) = \begin{cases} \binom{2n-1}{k} q^k (1-q)^{2n-1-k} & k = 0, 1, \ldots, 2n-1 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

(c) Since there are $K_n + 1$ winning tickets in all, the value of your winning ticket is $W_n = n/(K_n+1)$ which has mean

$$E[W_n] = nE\left[\frac{1}{K_n+1}\right] \tag{3}$$

Calculating the expected value

$$E\left[\frac{1}{K_n+1}\right] = \sum_{k=0}^{2n-1} \left(\frac{1}{k+1}\right) P_{K_n}(k) \tag{4}$$

is fairly complicated. The trick is to express the sum in terms of the sum of a binomial PMF.

$$E\left[\frac{1}{K_n+1}\right] = \sum_{k=0}^{2n-1} \frac{1}{k+1} \frac{(2n-1)!}{k!(2n-1-k)!} q^k (1-q)^{2n-1-k} \tag{5}$$

$$= \frac{1}{2n} \sum_{k=0}^{2n-1} \frac{(2n)!}{(k+1)!(2n-(k+1))!} q^k (1-q)^{2n-(k+1)} \tag{6}$$

By factoring out $1/q$, we obtain

$$E\left[\frac{1}{K_n+1}\right] = \frac{1}{2nq} \sum_{k=0}^{2n-1} \binom{2n}{k+1} q^{k+1} (1-q)^{2n-(k+1)} \tag{7}$$

$$= \frac{1}{2nq} \underbrace{\sum_{j=1}^{2n} \binom{2n}{j} q^j (1-q)^{2n-j}}_{A} \tag{8}$$

We observe that the above sum labeled A is the sum of a binomial PMF for $2n$ trials and success probability q over all possible values except $j = 0$. Thus

$$A = 1 - \binom{2n}{0} q^0 (1-q)^{2n-0} = 1 - (1-q)^{2n} \tag{9}$$

This implies

$$E\left[\frac{1}{K_n+1}\right] = \frac{1-(1-q)^{2n}}{2nq} \tag{10}$$

Our expected return on a winning ticket is

$$E\left[W_n\right] = nE\left[\frac{1}{K_n+1}\right] = \frac{1-(1-q)^{2n}}{2q} \tag{11}$$

Note that when $nq \ll 1$, we can use the approximation that $(1-q)^{2n} \approx 1-2nq$ to show that

$$E\left[W_n\right] \approx \frac{1-(1-2nq)}{2q} = n \qquad (nq \ll 1) \tag{12}$$

However, in the limit as the value of the prize n approaches infinity, we have

$$\lim_{n\to\infty} E\left[W_n\right] = \frac{1}{2q} \approx 4.683 \times 10^6 \tag{13}$$

That is, as the pot grows to infinity, the expected return on a winning ticket doesn't approach infinity because there is a corresponding increase in the number of other winning tickets. If it's not clear how large n must be for this effect to be seen, consider the following table:

n	10^6	10^7	10^8
$E\left[W_n\right]$	9.00×10^5	4.13×10^6	4.68×10^6

(14)

When the pot is \$1 million, our expected return is \$900,000. However, we see that when the pot reaches \$100 million, our expected return is very close to $1/(2q)$, less than \$5 million!

Problem 2.8.8 Solution

Given the following description of the random variable Y,

$$Y = \frac{1}{\sigma_x}(X-\mu_X) \tag{1}$$

we can use the linearity property of the expectation operator to find the mean value

$$E\left[Y\right] = \frac{E\left[X-\mu_X\right]}{\sigma_X} = \frac{E\left[X\right]-E\left[X\right]}{\sigma_X} = 0 \tag{2}$$

Using the fact that $\mathrm{Var}[aX+b] = a^2\,\mathrm{Var}[X]$, the variance of Y is found to be

$$\mathrm{Var}\left[Y\right] = \frac{1}{\sigma_X^2}\,\mathrm{Var}\left[X\right] = 1 \tag{3}$$

Problem 2.10.4 Solution

Suppose X_n is a Zipf $(n, \alpha = 1)$ random variable and thus has PMF

$$P_X(x) = \begin{cases} c(n)/x & x = 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The problem asks us to find the smallest value of k such that $P[X_n \leq k] \geq 0.75$. That is, if the server caches the k most popular files, then with $P[X_n \leq k]$ the request is for one of the k cached files. First, we might as well solve this problem for any probability p rather than just $p = 0.75$. Thus, in math terms, we are looking for

$$k = \min \left\{ k' | P\left[X_n \leq k'\right] \geq p \right\}. \tag{2}$$

What makes the Zipf distribution hard to analyze is that there is no closed form expression for

$$c(n) = \left(\sum_{x=1}^{n} \frac{1}{x} \right)^{-1}. \tag{3}$$

Thus, we use MATLAB to grind through the calculations. The following simple program generates the Zipf distributions and returns the correct value of k.

```
function k=zipfcache(n,p);
%Usage: k=zipfcache(n,p);
%for the Zipf (n,alpha=1) distribution, returns the smallest k
%such that the first k items have total probability p
pmf=1./(1:n);
pmf=pmf/sum(pmf); %normalize to sum to 1
cdf=cumsum(pmf);
k=1+sum(cdf<=p);
```

The program `zipfcache` generalizes 0.75 to be the probability p. Although this program is sufficient, the problem asks us to find k for all values of n from 1 to 10^3!. One way to do this is to call `zipfcache` a thousand times to find k for each value of n. A better way is to use the properties of the Zipf PDF. In particular,

$$P\left[X_n \leq k'\right] = c(n) \sum_{x=1}^{k'} \frac{1}{x} = \frac{c(n)}{c(k')} \tag{4}$$

Thus we wish to find

$$k = \min \left\{ k' | \frac{c(n)}{c(k')} \geq p \right\} = \min \left\{ k' | \frac{1}{c(k')} \geq \frac{p}{c(n)} \right\}. \tag{5}$$

Note that the definition of k implies that

$$\frac{1}{c(k')} < \frac{p}{c(n)}, \qquad k' = 1, \ldots, k-1. \tag{6}$$

Using the notation $|A|$ to denote the number of elements in the set A, we can write

$$k = 1 + \left| \left\{ k' | \frac{1}{c(k')} < \frac{p}{c(n)} \right\} \right| \tag{7}$$

This is the basis for a very short MATLAB program:

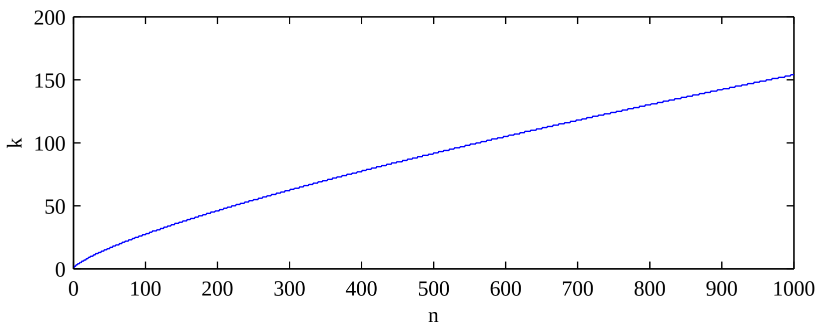
```
function k=zipfcacheall(n,p);
%Usage: k=zipfcacheall(n,p);
%returns vector k such that the first
%k(m) items have total probability >= p
%for the Zipf(m,1) distribution.
c=1./cumsum(1./(1:n));
k=1+countless(1./c,p./c);
```

Note that `zipfcacheall` uses a short MATLAB program `countless.m` that is almost the same as `count.m` introduced in Example 2.47. If `n=countless(x,y)`, then `n(i)` is the number of elements of `x` that are strictly less than `y(i)` while `count` returns the number of elements less than or equal to `y(i)`.

In any case, the commands

```
k=zipfcacheall(1000,0.75);
plot(1:1000,k);
```

is sufficient to produce this figure of k as a function of m:



We see in the figure that the number of files that must be cached grows slowly with the total number of files n.

Finally, we make one last observation. It is generally desirable for MATLAB to execute operations in parallel. The program `zipfcacheall` generally will run faster than n calls to `zipfcache`. However, to do its counting all at once, `countless` generates and $n \times n$ array. When n is not too large, say $n \leq 1000$, the resulting array with $n^2 = 1{,}000{,}000$ elements fits in memory. For much large values of n, say $n = 10^6$ (as was proposed in the original printing of this edition of the text, `countless` will cause an "out of memory" error.

Problem 3.2.5 Solution

$$f_X(x) = \begin{cases} ax^2 + bx & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

First, we note that a and b must be chosen such that the above PDF integrates to 1.

$$\int_0^1 (ax^2 + bx)\, dx = a/3 + b/2 = 1 \tag{2}$$

Hence, $b = 2 - 2a/3$ and our PDF becomes

$$f_X(x) = x(ax + 2 - 2a/3) \tag{3}$$

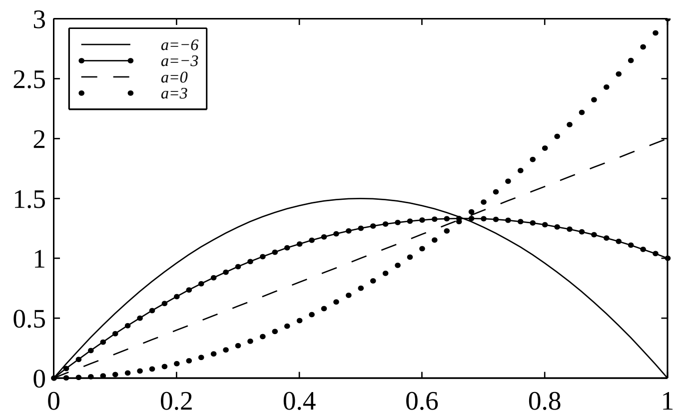
For the PDF to be non-negative for $x \in [0,1]$, we must have $ax + 2 - 2a/3 \geq 0$ for all $x \in [0,1]$. This requirement can be written as

$$a(2/3 - x) \leq 2 \qquad (0 \leq x \leq 1) \tag{4}$$

For $x = 2/3$, the requirement holds for all a. However, the problem is tricky because we must consider the cases $0 \leq x < 2/3$ and $2/3 < x \leq 1$ separately because of the sign change of the inequality. When $0 \leq x < 2/3$, we have $2/3 - x > 0$ and the requirement is most stringent at $x = 0$ where we require $2a/3 \leq 2$ or $a \leq 3$. When $2/3 < x \leq 1$, we can write the constraint as $a(x - 2/3) \geq -2$. In this case, the constraint is most stringent at $x = 1$, where we must have $a/3 \geq -2$ or $a \geq -6$. Thus a complete expression for our requirements are

$$-6 \leq a \leq 3 \qquad b = 2 - 2a/3 \tag{5}$$

As we see in the following plot, the shape of the PDF $f_X(x)$ varies greatly with the value of a.



Problem 3.4.14 Solution

(a) Since $f_X(x) \geq 0$ and $x \geq r$ over the entire integral, we can write

$$\int_r^\infty x f_X(x)\, dx \geq \int_r^\infty r f_X(x)\, dx = rP[X > r] \tag{1}$$

(b) We can write the expected value of X in the form

$$E[X] = \int_0^r x f_X(x)\, dx + \int_r^\infty x f_X(x)\, dx \tag{2}$$

Hence,

$$rP[X > r] \leq \int_r^\infty x f_X(x)\, dx = E[X] - \int_0^r x f_X(x)\, dx \tag{3}$$

Allowing r to approach infinity yields

$$\lim_{r \to \infty} rP[X > r] \leq E[X] - \lim_{r \to \infty} \int_0^r x f_X(x)\, dx = E[X] - E[X] = 0 \tag{4}$$

Since $rP[X > r] \geq 0$ for all $r \geq 0$, we must have $\lim_{r \to \infty} rP[X > r] = 0$.

(c) We can use the integration by parts formula $\int u\,dv = uv - \int v\,du$ by defining $u = 1 - F_X(x)$ and $dv = dx$. This yields

$$\int_0^\infty [1 - F_X(x)]\,dx = x[1 - F_X(x)]|_0^\infty + \int_0^\infty x f_X(x)\,dx \tag{5}$$

By applying part (a), we now observe that

$$x\,[1 - F_X(x)]|_0^\infty = \lim_{r\to\infty} r[1 - F_X(r)] - 0 = \lim_{r\to\infty} rP\,[X > r] \tag{6}$$

By part (b), $\lim_{r\to\infty} rP[X > r] = 0$ and this implies $x[1 - F_X(x)]|_0^\infty = 0$. Thus,

$$\int_0^\infty [1 - F_X(x)]\,dx = \int_0^\infty x f_X(x)\,dx = E\,[X] \tag{7}$$

Problem 3.5.10 Solution

This problem is mostly calculus and only a little probability. From the problem statement, the SNR Y is an exponential $(1/\gamma)$ random variable with PDF

$$f_Y(y) = \begin{cases} (1/\gamma)e^{-y/\gamma} & y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Thus, from the problem statement, the BER is

$$\overline{P}_e = E\,[P_e(Y)] = \int_{-\infty}^\infty Q(\sqrt{2y}) f_Y(y)\,dy = \int_0^\infty Q(\sqrt{2y}) \frac{y}{\gamma} e^{-y/\gamma}\,dy \tag{2}$$

Like most integrals with exponential factors, its a good idea to try integration by parts. Before doing so, we recall that if X is a Gaussian $(0,1)$ random variable with CDF $F_X(x)$, then

$$Q(x) = 1 - F_X(x)\,. \tag{3}$$

It follows that $Q(x)$ has derivative

$$Q'(x) = \frac{dQ(x)}{dx} = -\frac{dF_X(x)}{dx} = -f_X(x) = -\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{4}$$

To solve the integral, we use the integration by parts formula $\int_a^b u\,dv = uv|_a^b - \int_a^b v\,du$, where

$$u = Q(\sqrt{2y}) \qquad dv = \frac{1}{\gamma} e^{-y/\gamma}\,dy \tag{5}$$

$$du = Q'(\sqrt{2y}) \frac{1}{\sqrt{2y}} = -\frac{e^{-y}}{2\sqrt{\pi y}} \qquad v = -e^{-y/\gamma} \tag{6}$$

From integration by parts, it follows that

$$\overline{P}_e = uv|_0^\infty - \int_0^\infty v\,du = -Q(\sqrt{2y})e^{-y/\gamma}\Big|_0^\infty - \int_0^\infty \frac{1}{\sqrt{y}} e^{-y[1+(1/\gamma)]}\,dy \tag{7}$$

$$= 0 + Q(0)e^{-0} - \frac{1}{2\sqrt{\pi}} \int_0^\infty y^{-1/2} e^{-y/\bar{\gamma}}\,dy \tag{8}$$

where $\bar{\gamma} = \gamma/(1+\gamma)$. Next, recalling that $Q(0) = 1/2$ and making the substitution $t = y/\bar{\gamma}$, we obtain

$$\overline{P}_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\bar{\gamma}}{\pi}} \int_0^\infty t^{-1/2} e^{-t}\, dt \tag{9}$$

From Math Fact B.11, we see that the remaining integral is the $\Gamma(z)$ function evaluated $z = 1/2$. Since $\Gamma(1/2) = \sqrt{\pi}$,

$$\overline{P}_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\bar{\gamma}}{\pi}}\Gamma(1/2) = \frac{1}{2}\left[1 - \sqrt{\bar{\gamma}}\right] = \frac{1}{2}\left[1 - \sqrt{\frac{\gamma}{1+\gamma}}\right] \tag{10}$$

Problem 3.7.2 Solution

Since $Y = \sqrt{X}$, the fact that X is nonegative and that we asume the squre root is always positive implies $F_Y(y) = 0$ for $y < 0$. In addition, for $y \geq 0$, we can find the CDF of Y by writing

$$F_Y(y) = P[Y \leq y] = P\left[\sqrt{X} \leq y\right] = P\left[X \leq y^2\right] = F_X\left(y^2\right) \tag{1}$$

For $x \geq 0$, $F_X(x) = 1 - e^{-\lambda x}$. Thus,

$$F_Y(y) = \begin{cases} 1 - e^{-\lambda y^2} & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

By taking the derivative with respect to y, it follows that the PDF of Y is

$$f_Y(y) = \begin{cases} 2\lambda y e^{-\lambda y^2} & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

In comparing this result to the Rayleigh PDF given in Appendix A, we observe that Y is a Rayleigh (a) random variable with $a = \sqrt{2\lambda}$.

Problem 3.8.8 Solution

(a) The event B_i that $Y = \Delta/2 + i\Delta$ occurs if and only if $i\Delta \leq X < (i+1)\Delta$. In particular, since X has the uniform $(-r/2, r/2)$ PDF

$$f_X(x) = \begin{cases} 1/r & -r/2 \leq x < r/2, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

we observe that

$$P[B_i] = \int_{i\Delta}^{(i+1)\Delta} \frac{1}{r}\, dx = \frac{\Delta}{r} \tag{2}$$

In addition, the conditional PDF of X given B_i is

$$f_{X|B_i}(x) = \begin{cases} f_X(x)/P[B] & x \in B_i \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1/\Delta & i\Delta \leq x < (i+1)\Delta \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

It follows that given B_i, $Z = X - Y = X - \Delta/2 - i\Delta$, which is a uniform $(-\Delta/2, \Delta/2)$ random variable. That is,

$$f_{Z|B_i}(z) = \begin{cases} 1/\Delta & -\Delta/2 \leq z < \Delta/2 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(b) We observe that $f_{Z|B_i}(z)$ is the same for every i. Thus, we can write

$$f_Z(z) = \sum_i P[B_i] f_{Z|B_i}(z) = f_{Z|B_0}(z) \sum_i P[B_i] = f_{Z|B_0}(z) \tag{5}$$

Thus, Z is a uniform $(-\Delta/2, \Delta/2)$ random variable. From the definition of a uniform (a, b) random variable, Z has mean and variance

$$E[Z] = 0, \qquad \text{Var}[Z] = \frac{(\Delta/2 - (-\Delta/2))^2}{12} = \frac{\Delta^2}{12}. \tag{6}$$

Problem 3.8.9 Solution

For this problem, almost any non-uniform random variable X will yield a non-uniform random variable Z. For example, suppose X has the "triangular" PDF

$$f_X(x) = \begin{cases} 8x/r^2 & 0 \le x \le r/2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

In this case, the event B_i that $Y = i\Delta + \Delta/2$ occurs if and only if $i\Delta \le X < (i+1)\Delta$. Thus

$$P[B_i] = \int_{i\Delta}^{(i+1)\Delta} \frac{8x}{r^2}\, dx = \frac{8\Delta(i\Delta + \Delta/2)}{r^2} \tag{2}$$

It follows that the conditional PDF of X given B_i is

$$f_{X|B_i}(x) = \begin{cases} \frac{f_X(x)}{P[B_i]} & x \in B_i \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{x}{\Delta(i\Delta + \Delta/2)} & i\Delta \le x < (i+1)\Delta \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Given event B_i, $Y = i\Delta + \Delta/2$, so that $Z = X - Y = X - i\Delta - \Delta/2$. This implies

$$f_{Z|B_i}(z) = f_{X|B_i}(z + i\Delta + \Delta/2) = \begin{cases} \frac{z + i\Delta + \Delta/2}{\Delta(i\Delta + \Delta/2)} & -\Delta/2 \le z < \Delta/2 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

We observe that the PDF of Z depends on which event B_i occurs. Moreover, $f_{Z|B_i}(z)$ is non-uniform for all B_i.