Probability and Stochastic Processes:
A Friendly Introduction for Electrical and Computer Engineers
Edition 2
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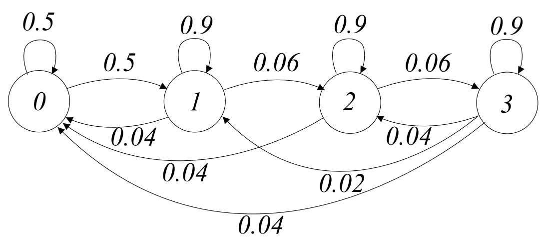
Problem Solutions : Yates and Goodman, 12.1.4 12.1.7 12.3.2 12.3.3 12.5.5 12.5.6 12.6.2 12.8.2 12.9.1 12.10.1 12.10.5 12.11.2 and 12.11.3

Problem 12.1.4 Solution

Based on the problem statement, the state of the wireless LAN is given by the following Markov chain:



The Markov chain has state transition matrix

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.04 & 0.9 & 0.06 & 0 \\ 0.04 & 0 & 0.9 & 0.06 \\ 0.04 & 0.02 & 0.04 & 0.9 \end{bmatrix}. \tag{1}$$

Problem 12.1.7 Solution

Let

$$\mathbf{Y} = \begin{bmatrix} Y_{n-1} & Y_{n-2} & \cdots & Y_0 \end{bmatrix}' = \begin{bmatrix} X_{T_{n-1}} & X_{T_{n-2}} & \cdots & X_0 \end{bmatrix}' \tag{1}$$

denote the past history of the process. In the conditional space where $Y_n = i$ and $\mathbf{Y} = \mathbf{y}$, we can use the law of total probability to write

$$\begin{aligned} &P[Y_{n+1} = j|Y_n = i, \mathbf{Y} = \mathbf{y}] \\ &\qquad = \sum_k P[Y_{n+1} = j, |Y_n = i, \mathbf{Y} = \mathbf{y}, K_n = k] P[K_n = k|Y_n = i, \mathbf{Y} = \mathbf{y}]. \end{aligned} \tag{2}$$

Since K_n is independent of Y_n and the past history $\mathbf{Y}$,

$$P[K_n = k|Y_n = i, \mathbf{Y} = \mathbf{y}] = P[K_n = k]. \tag{3}$$

Next we observe that

$$\begin{aligned} P[Y_{n+1} = j|Y_n = i, \mathbf{Y} = \mathbf{y}, K_n = k] &= P[X_{T_n+k} = j|X_{T_n} = i, K_n = k, \mathbf{Y} = \mathbf{y}] && (4) \\ &= P[X_{T_n+k} = j|X_{T_n} = i, K_n = k] && (5) \end{aligned}$$

because the state X_{T_n+k} is independent of the past history $\mathbf{Y}$ given the most recent state X_{T_n}. Moreover, by time invariance of the Markov chain,

$$P[X_{T_n+k} = j|X_{T_n} = i, K_n = k] = P[X_{T_n+k} = j|X_{T_n} = i] = [\mathbf{P}^k]_{ij}. \tag{6}$$

Equations (5) and (6) imply

$$P[Y_{n+1} = j|Y_n = i, \mathbf{Y} = \mathbf{y}, K_n = k] = [\mathbf{P}^k]_{ij}. \tag{7}$$

It then follows from Equation (2) that

$$P[Y_{n+1} = j|Y_n = i, \mathbf{Y} = \mathbf{y}] = \sum_k P[Y_{n+1} = j, |Y_n = i, \mathbf{Y} = \mathbf{y}, K_n = k]P[K_n = k] \tag{8}$$

$$= \sum_k [\mathbf{P}^k]_{ij} P[K_n = k]. \tag{9}$$

Thus $P[Y_{n+1} = j|Y_n = i, \mathbf{Y} = \mathbf{y}]$ depends on i and j and is independent of the past history $\mathbf{Y}$ and we conclude that Y_n is a Markov chain.

Problem 12.3.2 Solution

At time $n-1$, let $p_i(n-1)$ denote the state probabilities. By Theorem 12.4, the probability of state k at time n is

$$p_k(n) = \sum_{i=0}^{\infty} p_i(n-1) P_{ik} \tag{1}$$

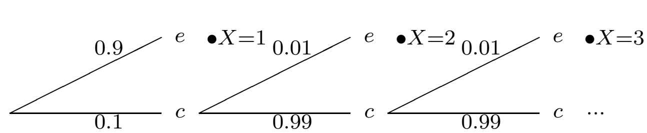
Since $P_{ik} = q$ for every state i,

$$p_k(n) = q \sum_{i=0}^{\infty} p_i(n-1) = q \tag{2}$$

Thus for any time $n > 0$, the probability of state k is q.

Problem 12.3.3 Solution

In this problem, the arrivals are the occurrences of packets in error. It would seem that $N(t)$ cannot be a renewal process because the interarrival times seem to depend on the previous interarrival times. However, following a packet error, the sequence of packets that are correct (c) or in error (e) up to and including the next error is given by the tree



Assuming that sending a packet takes one unit of time, the time X until the next packet error has the PMF

$$P_X(x) = \begin{cases} 0.9 & x = 1 \\ 0.001(0.99)^{x-2} & x = 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Thus, following an error, the time until the next error always has the same PMF. Moreover, this time is independent of previous interarrival times since it depends only on the Bernoulli trials following a packet error. It would appear that $N(t)$ is a renewal process; however, there is one additional complication. At time 0, we need to know the probability p of an error for the first packet. If $p = 0.9$, then X_1, the time until the first error, has the same PMF as X above and the process is a renewal process. If $p \neq 0.9$, then the time until the first error is different from subsequent renewal times. In this case, the process is a delayed renewal process.

Problem 12.5.5 Solution

For this system, it's hard to draw the entire Markov chain since from each state n there are six branches, each with probability $1/6$ to states $n+1, n+2, \ldots, n+6$. (Of course, if $n+k > K-1$, then the transition is to state $n+k \mod K$.) Nevertheless, finding the stationary probabilities is not very hard. In particular, the nth equation of $\boldsymbol{\pi}' = \boldsymbol{\pi}'\mathbf{P}$ yields

$$\pi_n = \frac{1}{6}\left(\pi_{n-6} + \pi_{n-5} + \pi_{n-4} + \pi_{n-3} + \pi_{n-2} + \pi_{n-1}\right). \tag{1}$$

Rather than try to solve these equations algebraically, it's easier to guess that the solution is

$$\boldsymbol{\pi} = \begin{bmatrix} 1/K & 1/K & \cdots & 1/K \end{bmatrix}'. \tag{2}$$

It's easy to check that $1/K = (1/6) \cdot 6 \cdot (1/K)$

Problem 12.5.6 Solution

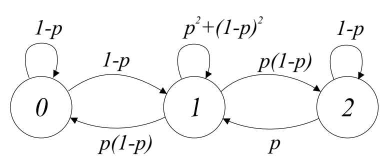
This system has three states:

0 front teller busy, rear teller idle

1 front teller busy, rear teller busy

2 front teller idle, rear teller busy

We will assume the units of time are seconds. Thus, if a teller is busy one second, the teller will become idle in th next second with probability $p = 1/120$. The Markov chain for this system is



We can solve this chain very easily for the stationary probability vector $\boldsymbol{\pi}$. In particular,

$$\pi_0 = (1-p)\pi_0 + p(1-p)\pi_1 \tag{1}$$

This implies that $\pi_0 = (1-p)\pi_1$. Similarly,

$$\pi_2 = (1-p)\pi_2 + p(1-p)\pi_1 \tag{2}$$

yields $\pi_2 = (1-p)\pi_1$. Hence, by applying $\pi_0 + \pi_1 + \pi_2 = 1$, we obtain

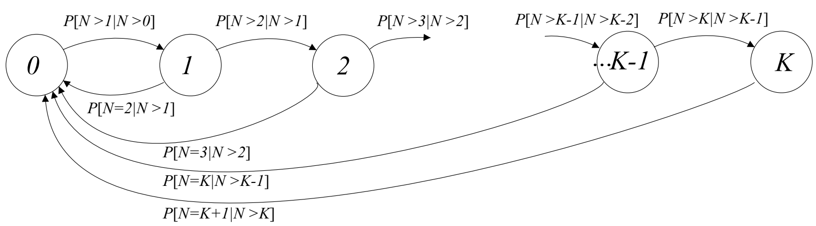
$$\pi_0 = \pi_2 = \frac{1-p}{3-2p} = 119/358 \tag{3}$$

$$\pi_1 = \frac{1}{3-2p} = 120/358 \tag{4}$$

The stationary probability that both tellers are busy is $\pi_1 = 120/358$.

Problem 12.6.2 Solution

The Markov chain for this system is



Note that $P[N > 0] = 1$ and that

$$P[N > n | N > n-1] = \frac{P[N > n, N > n-1]}{P[N > n-1]} = \frac{P[N > n]}{P[N > n-1]}. \tag{1}$$

Solving $\boldsymbol{\pi}' = \boldsymbol{\pi}'\mathbf{P}$ yields

$$\pi_1 = P[N > 1 | N > 0] \pi_0 = P[N > 1] \pi_0 \tag{2}$$

$$\pi_2 = P[N > 2 | N > 1] \pi_1 = \frac{P[N > 2]}{P[N > 1]} \pi_1 = P[N > 2] \pi_0 \tag{3}$$

$$\vdots$$

$$\pi_n = P[N > n | N > n-1] \pi_{n-1} = \frac{P[N > n]}{P[N > n-1]} \pi_{n-1} = P[N > n] \pi_0 \tag{4}$$

Next we apply the requirement that the stationary probabilities sum to 1. Since $P[N \leq K+1] = 1$, we see for $n \geq K+1$ that $P[N > n] = 0$. Thus

$$1 = \sum_{n=0}^{K} \pi_n = \pi_0 \sum_{n=0}^{K} P[N > n] = \pi_0 \sum_{n=0}^{\infty} P[N > n]. \tag{5}$$

From Problem 2.5.11, we recall that $\sum_{n=0}^{\infty} P[N > n] = E[N]$. This implies $\pi_0 = 1/E[N]$ and that

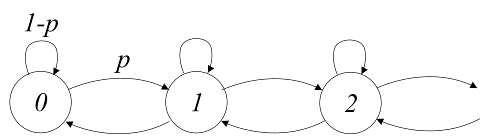
$$\pi_n = \frac{P[N > n]}{E[N]}. \tag{6}$$

This is exactly the same stationary distribution found in Quiz 12.5! In this problem, we can view the system state as describing the *age* of an object that is repeatedly replaced. In state 0, we start with a new (zero age) object, and each unit of time, the object ages one unit of time. The random variable N is the lifetime of the object. A transition to state 0 corresponds to the current object expiring and being replaced by a new object.

In Quiz 12.5, the system state described a countdown timer for the residual life of an object. At state 0, the system would transition to a state $N = n$ corresponding to the lifetime of n for a new object. This object expires and is replaced each time that state 0 is reached. This solution and the solution to Quiz 12.5 show that the age and the residual life have the same stationary distribution. That is, if we inspect an object at an arbitrary time in the distant future, the PMF of the age of the object is the same as the PMF of the residual life.

Problem 12.8.2 Solution

If there are k customers in the system at time n, then at time $n+1$, the number of customers in the system is either $n-1$ (if the customer in service departs and no new customer arrives), n (if either there is no new arrival and no departure *or* if there is both a new arrival and a departure) or $n+1$, if there is a new arrival but no new departure. The transition probabilities are given by the following chain:



where $\alpha = p(1-q)$ and $\delta = q(1-p)$. To find the stationary probabilities, we apply Theorem 12.13 by partitioning the state space between states $S = \{0, 1, \ldots, i\}$ and $S' = \{i+1, i+2, \ldots\}$ as shown in Figure 12.4. By Theorem 12.13, for state $i > 0$,

$$\pi_i \alpha = \pi_{i+1} \delta. \tag{1}$$

This implies $\pi_{i+1} = (\alpha/\delta)\pi_i$. A cut between states 0 and 1 yields $\pi_1 = (p/\delta)\pi_0$. Combining these results, we have for any state $i > 0$,

$$\pi_i = \frac{p}{\delta}\left(\frac{\alpha}{\delta}\right)^{i-1} \pi_0 \tag{2}$$

Under the condition $\alpha < \delta$, it follows that

$$\sum_{i=0}^{\infty} \pi_i = \pi_0 + \pi_0 \sum_{i=1}^{\infty} \frac{p}{\delta}\left(\frac{\alpha}{\delta}\right)^{i-1} = \pi_0 \left(1 + \frac{p/\delta}{1-\alpha/\delta}\right) \tag{3}$$

since $p < q$ implies $\alpha/\delta < 1$. Thus, applying $\sum_i \pi_i = 1$ and noting $\delta - \alpha = q - p$, we have

$$\pi_0 = \frac{q}{q-p}, \qquad \pi_i = \frac{p}{(1-p)(1-q)} \left[\frac{p/(1-p)}{q/(1-q)}\right]^{i-1}, \quad i = 1, 2, \ldots \tag{4}$$

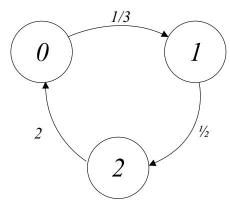
Note that $\alpha < \delta$ if and only if $p < q$, which is both sufficient and necessary for the Markov chain to be positive recurrent.

Problem 12.9.1 Solution

From the problem statement, we learn that in each state i, the tiger spends an exponential time with parameter λ_i. When we measure time in hours,

$$\lambda_0 = q_{01} = 1/3 \qquad \lambda_1 = q_{12} = 1/2 \qquad \lambda_2 = q_{20} = 2 \tag{1}$$

The corresponding continous time Markov chain is shown below:



The state probabilities satisfy

$$\frac{1}{3}p_0 = 2p_2 \qquad \frac{1}{2}p_1 = \frac{1}{3}p_0 \qquad p_0 + p_1 + p_2 = 1 \tag{2}$$

The solution is

$$\begin{bmatrix} p_0 & p_1 & p_2 \end{bmatrix} = \begin{bmatrix} 6/11 & 4/11 & 1/11 \end{bmatrix} \tag{3}$$

Problem 12.10.1 Solution

In Equation (12.93), we found that the blocking probability of the $M/M/c/c$ queue was given by the Erlang-B formula

$$P[B] = P_N(c) = \frac{\rho^c/c!}{\sum_{k=0}^{c} \rho^k/k!} \tag{1}$$

The parameter $\rho = \lambda/\mu$ is the normalized load. When $c = 2$, the blocking probability is

$$P[B] = \frac{\rho^2/2}{1 + \rho + \rho^2/2} \tag{2}$$

Setting $P[B] = 0.1$ yields the quadratic equation

$$\rho^2 - \frac{2}{9}\rho - \frac{2}{9} = 0 \tag{3}$$

The solutions to this quadratic are

$$\rho = \frac{1 \pm \sqrt{19}}{9} \tag{4}$$

The meaningful nonnegative solution is $\rho = (1 + \sqrt{19})/9 = 0.5954$.

Problem 12.10.5 Solution

(a) In this case, we have two $M/M/1$ queues, each with an arrival rate of $\lambda/2$. By defining $\rho = \lambda/\mu$, each queue has a stationary distribution

$$p_n = (1-\rho/2)\,(\rho/2)^n \qquad n = 0,1,\ldots \tag{1}$$

Note that in this case, the expected number in queue i is

$$E[N_i] = \sum_{n=0}^{\infty} n p_n = \frac{\rho/2}{1-\rho/2} \tag{2}$$

The expected number in the system is

$$E[N_1] + E[N_2] = \frac{\rho}{1-\rho/2} \tag{3}$$

(b) The combined queue is an $M/M/2/\infty$ queue. As in the solution to Quiz 12.10, the stationary probabilities satisfy

$$p_n = \begin{cases} p_0\rho^n/n! & n = 1,2 \\ p_0\rho^{n-2}\rho^2/2 & n = 3,4,\ldots \end{cases} \tag{4}$$

The requirement that $\sum_{n=0}^{\infty} p_n = 1$ yields

$$p_0 = \left(1 + \rho + \frac{\rho^2}{2} + \frac{\rho^2}{2}\frac{\rho/2}{1-\rho/2}\right)^{-1} = \frac{1-\rho/2}{1+\rho/2} \tag{5}$$

The expected number in the system is $E[N] = \sum_{n=1}^{\infty} n p_n$. Some algebra will show that

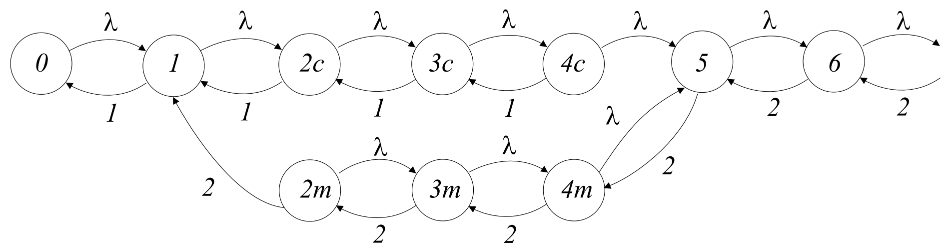
$$E[N] = \frac{\rho}{1-(\rho/2)^2} \tag{6}$$

We see that the average number in the combined queue is lower than in the system with individual queues. The reason for this is that in the system with individual queues, there is a possibility that one of the queues becomes empty while there is more than one person in the other queue.

Problem 12.11.2 Solution

In this problem, we model the system as a continuous time Markov chain. The clerk and the manager each represent a "server." The state describes the number of customers in the queue and the number of active servers. The Markov chain issomewhat complicated because when the number of customers in the store is 2, 3, or 4, the number of servers may be 1 or may be 2, depending on whether the manager became an active server.

When just the clerk is serving, the service rate is 1 customer per minute. When the manager and clerk are both serving, the rate is 2 customers per minute. Here is the Markov chain:



In states $2c$, $3c$ and $4c$, only the clerk is working. In states $2m$, $3m$ and $4m$, the manager is also working. The state space $\{0, 1, 2c, 3c, 4c, 2m, 3m, 4m, 5, 6, \ldots\}$ is countably infinite. Finding the state probabilities is a little bit complicated because the are enough states that we would like to use MATLAB; however, MATLAB can only handle a finite state space. Fortunately, we can use MATLAB because the state space for states $n \geq 5$ has a simple structure.

We observe for $n \geq 5$ that the average rate of transitions from state n to state $n+1$ must equal the average rate of transitions from state $n+1$ to state n, implying

$$\lambda p_n = 2p_{n+1}, \qquad n = 5, 6, \ldots \tag{1}$$

It follows that $p_{n+1} = (\lambda/2)p_n$ and that

$$p_n = \alpha^{n-5} p_5, \qquad n = 5, 6, \ldots, \tag{2}$$

where $\alpha = \lambda < 2 < 1$. The requirement that the stationary probabilities sum to 1 implies

$$1 = p_0 + p_1 + \sum_{j=2}^{4}(p_{jc} + p_{jm}) + \sum_{n=5}^{\infty} p_n \tag{3}$$

$$= p_0 + p_1 + \sum_{j=2}^{4}(p_{jc} + p_{jm}) + p_5 \sum_{n=5}^{\infty} \alpha^{n-5} \tag{4}$$

$$= p_0 + p_1 + \sum_{j=2}^{4}(p_{jc} + p_{jm}) + \frac{p_5}{1-\alpha} \tag{5}$$

This is convenient because for each state $j < 5$, we can solve for the staitonary probabilities. In particular, we use Theorem 12.23 to write $\sum \sum_i r_{ij} p_i = 0$. This leads to a set of matrix equations for the state probability vector

$$\mathbf{p} = \begin{bmatrix} p_0 & p_1 & p_{2c} & p_{3c} & p_{3c} & p_{4c} & p_{2m} & p_{3m} & p_{4m} & p_5 \end{bmatrix}' \tag{6}$$

The rate transition matrix associated with $\mathbf{p}$ is

$$\mathbf{Q} = \left[\begin{array}{ccccccccc}
p_0 & p_1 & p_{2c} & p_{3c} & p_{4c} & p_{2m} & p_{3m} & p_{4m} & p_5 \\
\hline
0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \lambda & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \lambda \\
0 & 2 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & \lambda & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & \lambda \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0
\end{array}\right], \tag{7}$$

where the first row just shows the correspondence of the state probabilities and the matrix columns. For each state i, excepting state 5, the departure rate ν_i from that state equals the sum of entries of the corresponding row of $\mathbf{Q}$. To find the stationary probabilities, our normal procedure is to use Theorem 12.23 and solve $\mathbf{p}'\mathbf{R} = \mathbf{0}$ and $\mathbf{p}'\mathbf{1} = 1$, where $\mathbf{R}$ is the same as $\mathbf{Q}$ except the zero diagonal entries are replaced by $-\nu_i$. The equation $\mathbf{p}'\mathbf{1} = 1$ replaces one column of the set of matrix equations. This is the approach of `cmcstatprob.m`.

In this problem, we follow almost the same procedure. We form the matrix $\mathbf{R}$ by replacing the diagonal entries of $\mathbf{Q}$. However, instead of replacing an arbitrary column with the equation $\mathbf{p}'\mathbf{1} = 1$, we replace the column corresponding to p_5 with the equation

$$p_0 + p_1 + p_{2c} + p_{3c} + p_{4c} + p_{2m} + p_{3m} + p_{4m} + \frac{p_5}{1-\alpha} = 1. \tag{8}$$

That is, we solve

$$\mathbf{p}'\mathbf{R} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}'. \tag{9}$$

where

$$\mathbf{R} = \begin{bmatrix}
-\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & -1-\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & -1-\lambda & \lambda & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & -1-\lambda & \lambda & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & -1-\lambda & 0 & 0 & 0 & 1 \\
0 & 2 & 0 & 0 & 0 & -2-\lambda & \lambda & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 2 & -2-\lambda & \lambda & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & -2-\lambda & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & \frac{1}{1-\alpha}
\end{bmatrix} \tag{10}$$

Given the stationary distribution, we can now find $E[N]$ and $P[W]$.

Recall that N is the number of customers in the system at a time in the distant future. Defining

$$p_n = p_{nc} + p_{nm}, \qquad n = 2, 3, 4, \tag{11}$$

we can write

$$E[N] = \sum_{n=0}^{\infty} n p_n = \sum_{n=0}^{4} n p_n + \sum_{n=5}^{\infty} n p_5 \alpha^{n-5} \tag{12}$$

The substitution $k = n - 5$ yields

$$E[N] = \sum_{n=0}^{4} np_n + p_5 \sum_{k=0}^{\infty} (k+5)\alpha^k \tag{13}$$

$$= \sum_{n=0}^{4} np_n + p_5 \frac{5}{1-\alpha} + p_5 \sum_{k=0}^{\infty} k\alpha^k \tag{14}$$

From Math Fact B.7, we conclude that

$$E[N] = \sum_{n=0}^{4} np_n + p_5 \left(\frac{5}{1-\alpha} + \frac{\alpha}{(1-\alpha)^2} \right) \tag{15}$$

$$= \sum_{n=0}^{4} np_n + p_5 \frac{5-4\alpha}{(1-\alpha)^2} \tag{16}$$

Furthermore, the manager is working unless the system is in state 0, 1, $2c$, $3c$, or $4c$. Thus

$$P[W] = 1 - (p_0 + p_1 + p_{2c} + p_{3c} + p_{4c}). \tag{17}$$

We implement these equations in the following program, alongside the corresponding output.

```
function [EN,PW]=clerks(lam);
Q=diag(lam*[1 1 1 1 0 1 1 1],1);
Q=Q+diag([1 1 1 1 0 2 2 2],-1);
Q(6,2)=2; Q(5,9)=lam;
R=Q-diag(sum(Q,2));
n=size(Q,1);
a=lam/2;
R(:,n)=[ones(1,n-1) 1/(1-a)]';
pv=([zeros(1,n-1) 1]*R^(-1));
EN=pv*[0;1;2;3;4;2;3;4; ...
        (5-4*a)/(1-a)^2];
PW=1-sum(pv(1:5));
```

```
>> [en05,pw05]=clerks(0.5)
en05 =
    0.8217
pw05 =
    0.0233
>> [en10,pw10]=clerks(1.0)
en10 =
    2.1111
pw10 =
    0.2222
>> [en15,pw15]=clerks(1.5)
en15 =
    4.5036
pw15 =
    0.5772
>>
```

We see that in going from an arrival rate of 0.5 customers per minute to 1.5 customers per minute, the average number of customers goes from 0.82 to 4.5 customers. Similarly, the probability the manager is working rises from 0.02 to 0.57.

Problem 12.11.3 Solution

Although the inventory system in this problem is relatively simple, the performance analysis is suprisingly complicated. We can model the system as a Markov chain with state X_n equal to the number of brake pads in stock at the start of day n. Note that the range of X_n is $S_X = \{50, 51, \ldots, 109\}$. To evaluate the system, we need to find the state transition matrix for the Markov chain. We express the transition probabilities in terms of $P_K(\cdot)$, the PMF of the number of brake pads ordered on an arbitary day. In state i, there are two possibilities:

- If $50 \le i \le 59$, then there will be $\min(i, K)$ brake pads sold. At the end of the day, the number of pads remaining is less than 60, and so 50 more pads are delivered overnight. Thus the next state is $j = 50$ if $K \ge i$ pads are ordered, i pads are sold and 50 pads are delivered overnight. On the other hand, if there are $K < i$ orders, then the next state is $j = i - K + 50$. In this case,

$$P_{ij} = \begin{cases} P\left[K \ge i\right] & j = 50, \\ P_K\left(50 + i - j\right) & j = 51, 52, \ldots, 50 + i. \end{cases} \tag{1}$$

- If $60 \le i \le 109$, then there are several subcases:
  - $j = 50$: If there are $K \ge i$ orders, then all i pads are sold, 50 pads are delivered overnight, and the next state is $j = 50$. Thus

    $$P_{ij} = P\left[K \ge i\right], \qquad j = 50. \tag{2}$$

  - $51 \le j \le 59$: If $50 + i - j$ pads are sold, then $j - 50$ pads ar left at the end of the day. In this case, 50 pads are delivered overnight, and the next state is j with probability

    $$P_{ij} = P_K\left(50 + i - j\right), \qquad j = 51, 52, \ldots, 59. \tag{3}$$

  - $60 \le j \le i$: If there are $K = i - j$ pads ordered, then there will be $j \ge 60$ pads at the end of the day and the next state is j. On the other hand, if $K = 50 + i - j$ pads are ordered, then there will be $i - (50 + i - j) = j - 50$ pads at the end of the day. Since $60 \le j \le 109$, $10 \le j - 50 \le 59$, there will be 50 pads delivered overnight and the next state will be j. Thus

    $$P_{ij} = P_K\left(i - j\right) + P_K\left(50 + i - j\right), \qquad j = 60, 61, \ldots, i. \tag{4}$$

  - For $i < j \le 109$, state j can be reached from state i if there $50 + i - j$ orders, leaving $i - (50 + i - j) = j - 50$ in stock at the end of the day. This implies 50 pads are delivered overnight and the next stage is j. the probability of this event is

    $$P_{ij} = P_K\left(50 + i - j\right), \qquad j = i + 1, i + 2, \ldots, 109. \tag{5}$$

We can summarize these observations in this set of state transition probabilities:

$$P_{ij} = \begin{cases} P\left[K \ge i\right] & 50 \le i \le 109, j = 50, \\ P_K\left(50 + i - j\right) & 50 \le i \le 59, 51 \le j \le 50 + i, \\ P_K\left(50 + i - j\right) & 60 \le i \le 109, 51 \le j \le 59, \\ P_K\left(i - j\right) + P_K\left(50 + i - j\right) & 60 \le i \le 109, 60 \le j \le i, \\ P_K\left(50 + i - j\right) & 60 \le i \le 108, i + 1 \le j \le 109 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Note that the "0 otherwise" rule comes into effect when $50 \le i \le 59$ and $j > 50 + i$. To simplify these rules, we observe that $P_K(k) = 0$ for $k < 0$. This implies $P_K(50 + i - j) = 0$

for $j > 50 + i$. In addition, for $j > i$, $P_K(i - j) = 0$. These facts imply that we can write the state transition probabilities in the simpler form:

$$P_{ij} = \begin{cases} P\left[K \geq i\right] & 50 \leq i \leq 109, j = 50, \\ P_K\left(50 + i - j\right) & 50 \leq i \leq 59, 51 \leq j \\ P_K\left(50 + i - j\right) & 60 \leq i \leq 109, 51 \leq j \leq 59, \\ P_K\left(i - j\right) + P_K\left(50 + i - j\right) & 60 \leq i \leq 109, 60 \leq j \end{cases} \tag{7}$$

Finally, we make the definitions

$$\beta_i = P\left[K \geq i\right], \qquad \gamma_k = P_K\left(50 + k\right), \qquad \delta_k = P_K\left(k\right) + P_K\left(50 + k\right). \tag{8}$$

With these definitions, the state transition probabilities are

$$P_{ij} = \begin{cases} \beta_i & 50 \leq i \leq 109, j = 50, \\ \gamma_{i-j} & 50 \leq i \leq 59, 51 \leq j \\ \gamma_{i-j} & 60 \leq i \leq 109, 51 \leq j \leq 59, \\ \delta_{i-j} & 60 \leq i \leq 109, 60 \leq j \end{cases} \tag{9}$$

Expressed as a table, the state transition matrix $\mathbf{P}$ is

$$\begin{array}{c|cccccccccc}
i \backslash j & 50 & 51 & \cdots & 59 & 60 & \cdots & \cdots & \cdots & \cdots & 109 \\ \hline
50 & \beta_{50} & \gamma_{-1} & \cdots & \gamma_{-9} & \cdots & \cdots & \cdots & \cdots & \cdots & \gamma_{-59} \\
51 & \beta_{51} & \gamma_0 & \ddots & \vdots & \ddots & & & & & \vdots \\
\vdots & \vdots & \vdots & \ddots & \gamma_{-1} & & \ddots & & & \ddots & \vdots \\
59 & \beta_{59} & \gamma_8 & \cdots & \gamma_0 & \gamma_{-1} & \cdots & \gamma_{-9} & \cdots & \cdots & \gamma_{-50} \\
60 & \beta_{60} & \gamma_9 & \cdots & \gamma_1 & \delta_0 & \cdots & & \delta_{-9} & \cdots & \delta_{-49} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & & & \ddots & \vdots \\
\vdots & \vdots & \vdots & & \gamma_9 & \vdots & & & & & \delta_{-9} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \delta_9 & & & & & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & & & \ddots & \vdots \\
109 & \beta_{109} & \gamma_{58} & \cdots & \gamma_{50} & \delta_{49} & \cdots & \delta_9 & \cdots & \cdots & \delta_0
\end{array} \tag{10}$$

In terms of Matlab, all we need to do is to encode the matrix $\mathbf{P}$, calculate the stationary probability vector $\boldsymbol{\pi}$, and then calculate $E[Y]$, the expected number of pads sold on a typical day. To calculate $E[Y]$, we use iterated expectation. The number of pads ordered is the Poisson random variable K. We assume that on a day n that $X_n = i$ and we calculate the conditional expectation

$$E\left[Y|X_n = i\right] = E\left[\min(K, i)\right] = \sum_{j=0}^{i-1} j P_K\left(j\right) + i P\left[K \geq i\right]. \tag{11}$$

Since only states $i \geq 50$ are possible, we can write

$$E\left[Y|X_n = i\right] = \sum_{j=0}^{48} j P_K\left(j\right) + \sum_{j=49}^{i-1} j P_K\left(j\right) + i P\left[K \geq i\right]. \tag{12}$$

Finally, we assume that on a typical day n, the state of the system X_n is described by the stationary probabilities $P[X_n = i] = \pi_i$ and we calculate

$$E[Y] = \sum_{i=50}^{109} E[Y|X_n = i]\,\pi_i. \tag{13}$$

These calculations are given in this MATLAB program:

```
function [pstat,ey]=brakepads(alpha);
s=(50:109)';
beta=1-poissoncdf(alpha,s-1);
grow=poissonpmf(alpha,50+(-1:-1:-59));
gcol=poissonpmf(alpha,50+(-1:58));
drow=poissonpmf(alpha,0:-1:-49);
dcol=poissonpmf(alpha,0:49);
P=[beta,toeplitz(gcol,grow)];
P(11:60,11:60)=P(11:60,11:60)...
    +toeplitz(dcol,drow);
pstat=dmcstatprob(P);
[I,J]=ndgrid(49:108,49:108);
G=J.*(I>=J);
EYX=(G*gcol)+(s.*beta);
pk=poissonpmf(alpha,0:48);
EYX=EYX+(0:48)*pk;
ey=(EYX')*pstat;
```

The first half of `brakepads.m` constructs $\mathbf{P}$ to calculate the stationary probabilities. The first column of $\mathbf{P}$ is just the vector

$$\texttt{beta} = \begin{bmatrix}\beta_{50} & \cdots & \beta_{109}\end{bmatrix}'. \tag{14}$$

The rest of $\mathbf{P}$ is easy to construct using `toeplitz` function. We first build an asymmetric Toeplitz matrix with first row and first column

$$\texttt{grow} = \begin{bmatrix}\gamma_{-1} & \gamma_{-2} & \cdots & \gamma_{-59}\end{bmatrix} \tag{15}$$

$$\texttt{gcol} = \begin{bmatrix}\gamma_{-1} & \gamma_{0} & \cdots & \gamma_{58}\end{bmatrix}' \tag{16}$$

Note that $\delta_k = P_K(k) + \gamma_k$. Thus, to construct the Toeplitz matrix in the lower right corner of $\mathbf{P}$, we simply add the Toeplitz matrix corresponding to the missing $P_K(k)$ term. The second half of `brakepads.m` calculates $E[Y]$ using the iterated expectation. Note that

$$\texttt{EYX} = \begin{bmatrix}E[Y|X_n = 50] & \cdots & E[Y|X_n = 109]\end{bmatrix}'. \tag{17}$$

The somewhat convoluted code becomes clearer by noting the following correspondences:

$$E[Y|X_n = i] = \underbrace{\sum_{j=0}^{48} jP_K(j)}_{\texttt{(0:48)*pk}} + \underbrace{\sum_{j=49}^{i-1} jP_K(j)}_{\texttt{G*gcol}} + \underbrace{iP[K \geq i]}_{\texttt{s.*beta}}. \tag{18}$$

To find $E[Y]$, we execute the commands:

```
>> [ps,ey]=brakepads(50);
>> ey
ey =
   49.4154
>>
```

We see that although the store receives 50 orders for brake pads on average, the average number sold is only 49.42 because once in awhile the pads are out of stock. Some experimentation will show that if the expected number of orders α is significantly less than 50, then the expected number of brake pads sold each days is very close to α. On the other hand, if $\alpha \gg 50$, then the each day the store will run out of pads and will get a delivery of 50 pads ech night. The expected number of unfulfilled orders will be very close to $\alpha - 50$.

Note that a new inventory policy in which the overnight delivery is more than 50 pads or the threshold for getting a shipment is more than 60 will reduce the expected numer of unfulfilled orders. Whether such a change in policy is a good idea depends on factors such as the carrying cost of inventory that are absent from our simple model.