332:541 Stochastic Signals and Systems Fall 2007

Monte Hall Pop Quiz

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1. You are a contestant on a game show and are shown three different doors. Behind two doors are undesirable prizes and behind one door is a great prize. The great prize is equally likely to be behind any of the doors. You don't know behind which door resides which prize, but the game show host, Monte Hall, does. The game proceeds as follows:

   - You are asked to choose a door
   - Whatever door you pick, Monte then opens a door which has a goat behind it
   - You are allowed to change your choice (or not)
   - Your prize is revealed

   Should you change doors after Monte opens a goat door?

   Use a probability tree to formally justify your yes/no answer.

   SOLUTION: *If you stick with your door no matter what, your probability of winning is 1/3 – it doesn't matter what Monte says to you.*

   Now, suppose you're going to make a choice, stick or stay. Let's do a simple thought experiment and look at a randomization of your choice – you stick or stay with probability 1/2. Well, you choose a door and Monte opens a goat door (either one of two you didn't pick or the other goat door from the one you picked). You then choose a new door from the remaining two with probability 1/2. For this overall experiment, it does not matter what door you pick first because your final door is going to be a choice between a goat door and a prize door because Monte ALWAYS removes one of the goats.

   So your probability of winning this RANDOMIZED experiment is 1/2 – which, by the way, is greater than the 1/3 you get if you ALWAYS stick and essentially ignore Monte. Since choosing a new door at random is a superset of the "always switch" or "always stick" strategy (just adjust the pick probabilities away from 1/2 each toward stick or switch) we KNOW that some changing of doors is a good idea. The question is now of how much. That is, always sticking is worse than re-choosing at random after Monte reveals a goat.

   We walk down the probability branches.

   (a) *Pick* G_1 *(1/3)*
      i. *Always Stick* $\rightarrow$ *you lose (1)*
      ii. *Always Switch* $\rightarrow$ *you WIN (1)*

   (b) *Pick* G_2 *(1/3)*
      i. *Always Stick* $\rightarrow$ *you lose (1)*
      ii. *Always Switch* $\rightarrow$ *you WIN (1)*

   (c) *Pick* **PRIZE** *(1/3)*
      i. *Always Stick* $\rightarrow$ *you WIN (1)*
      ii. *Always Switch* $\rightarrow$ *you lose (1)*

The numbers in the parentheses are the probabilities associated with the relevant event. Looking at the "Always Stick" decision we see that there are three terminal positions – two losses and a win. All are equally likely, so the probability you LOSE if you stick is 2/3. Conversely, for the "Always Switch" strategy, there are two wins and one loss. So the probability you WIN with "Always Switch" is 2/3.

"Always Switch" is a better strategy than "Always Stick."

Now, taking the problem a little farther, suppose you switch with probability p. Then we have

(a) *Pick G_1 (1/3)*
   i. *You stick and lose (1-p)*
   ii. *You swtich and WIN (p)*
(b) *Pick G_2 (1/3)*
   i. *You stick and lose (1-p)*
   ii. *You swtich and WIN (p)*
(c) *Pick* **PRIZE** *(1/3)*
   i. *You stick and WIN (1-p)*
   ii. *You swtich and lose (p)*

The probability of winning is

$$\textit{Prob(WIN)} = (1/3)p + (1/3)p + (1/3)(1-p) = (1+p)/3$$

Clearly you maximize your probability of winning by choosing $p = 1$.

The point of this problem? Learn to use probability trees. Often the solution to a problem is looking at the event space and finding the right probability tree representation.