THE STATE UNIVERSITY OF NEW JERSEY

RUTGERS

College of Engineering
Department of Electrical and Computer Engineering

332:541 **Stochastic Signals and Systems** **Fall 2007**

Quiz I

There are 3 questions. You have the class period to answer them. Show all work. Answers given without work will receive no credit. GOOD LUCK!

Useful facts:

$\sum_{k=0}^{K} z^k = \frac{1-z^{K+1}}{1-z}$ $(z \neq 1)$	$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$	$E[X] = \int_0^\infty (1 - F_X(x))dx$ X non-negative

1. *(30 points)* Rutgera Univera, the world famous Rutgers University ECE graduate student has opened a painting business called High Probability Coverage (HPC). Her workers are armed with paintball guns which they shoot at the object to be painted. Trying to cut costs, Rutgera bought used paintball guns and their triggers are a bit unpredictable. If you hold the trigger down, paint balls are shot, but the times T_i in seconds between shots i and $i+1$ are mutually independent exponential random variable with mean $1/\lambda$. That is $f_{T_i}(t_i) = \lambda e^{-\lambda t_i}$.

   Rutgera has trained her workers to paint in straight lines by sweeping the muzzle of the gun at a constant velocity v in distance-units per second. When a paint ball hits the target, it paints a unit-radius splotch.

   (a) *(10 points)* What is the distribution on the distance L_i between the centers of paint ball splotches i and $i+1$?

   (b) *(10 points)* What is the probability that any given two successive splotches will not overlap?

   (c) *(10 points)* What is the probability that a sequence of K splotches will overlap but the $(K+1)$st splotch will leave a gap?

2. *(35 points)* We glossed over the mechanics of the central limit theorem in class. You're going to explore them here.

   (a) *(5 points)* The moment generating function of a random variable X is

   $$\Phi_X(s) = \int_{-\infty}^{\infty} e^{sx} f_X(x)dx$$

   It's three term Taylor series about $s = 0$ is $\Phi_X(s) \approx a + bs + c\frac{s^2}{2}$. Assuming X is zero mean and has variance σ^2, please determine a, b, and c.

(b) *(10 points)* We form the random variable $W_n = \frac{1}{\sqrt{n\sigma^2}} \sum_{i=1}^{n} X_i$ where the X_i are mutually independent and each has distribution $f_X(x)$. What is $E[W_n]$? What is $E[W_n^2]$? Write down an exact expression for $\Phi_{W_n}()$ in terms of $\Phi_X()$. BE CAREFUL.

(c) *(5 points)* Write down an approximation for $\Phi_{W_n}(s)$ in terms of the Taylor approximation for $\Phi_X(s)$.

(d) *(10 points)* Show that $\lim_{n\to\infty}(1+\frac{z}{n})^n = e^z$.

(e) *(5 points)* What is the limit as $n \to \infty$ for the Taylor approximation to $\Phi_{W_n}(s)$?

3. *(35 points)* Every morning, a large family of very sophisticated wombats chooses D from a geometric distribution $p_D(d) = p(1-p)^{d-1}$, $d = 1, 2, \ldots$. D is the number wombats who will make that morning's perilous journey across a busy road to bring back food. Cars come down the road with an average frequency of λ cars per second. The interarrival time Δ_i between successive cars is an exponential random variable with mean $1/\lambda$. That is $f_\Delta(\delta) = \lambda e^{-\lambda\delta}$. The Δ_i are mutually independent.

   The time T it takes any given wombat to cross the road is also an exponential random variable but with different mean $1/\kappa$. The T_i are assumed mutually independent. In addition, it is assumed that the $\{T_i\}$ and $\{\Delta_i\}$ are also mutually independent.

   Any wombats in the road when a car comes along are instantly killed.

   (a) *(10 points)* Show that $\Phi_T(s) = \frac{\kappa}{\kappa - s}$ and that $\Phi_D(s) = \frac{pe^s}{1-(1-p)e^s}$.

   (b) *(10 points)* Let the random variable W be the time required for all the wombats to get across the road. Write down an analytic expression for the probability that all wombats survive in terms of $f_W(w)$ and $f_\Delta(\delta)$. Simplify your result in terms of $\Phi_W()$.

   (c) *(10 points)* Suppose the wombats cross sequentially (the next one starts just after the one ahead reaches the other side). What is the probability that they all survive in terms of p, κ and λ?

   (d) *(5 points)* What value of p maximizes the probability that no wombats are killed?