RUTGERS

Department of Electrical & Computer Engineering

332:541 **Stochastic Signals and Systems** **Fall 2007**

Quiz III

There are 4 questions. You have three hours to answer them. Show all work. Answers given without work will receive no credit. GOOD LUCK!

1. *(35 points)* Let k be a parameter which can take on values $k = 0, 1, \cdots$. We form a random variable R as

$$R = \sum_{\ell=1}^{k^2} G_\ell$$

where the G_ℓ are i.i.d. zero mean unit variance Gaussian random variables. If $k = 0$ we define $R = 0$

   (a) *(5 points)* Given k, what is the probability density for R?

   (b) *(10 points)* What is the ML estimate for k, $\hat{k}(R)$? What is $E[\hat{k}(R)|k]$? Is this estimate biased or unbiased?

   (c) *(10 points)* What is the ML estimate for k^2, $\hat{k}^2(R)$? What is $E[\hat{k}^2(R)|k]$? Is this estimate biased or unbiased?

   (d) *(10 points)* Suppose we have a series of identically composed, but independent measurements R_m where $m = 1, 2, \cdots, M$. What is the ML estimate for k^2 based on these M measurements? Is the estimate biased or unbiased? Is the estimate consistent?

2. *(35 points)* Suppose Y and X are zero mean jointly Gaussian random variables and you wish to estimate Y from X. If we define **Z** as

$$\mathbf{Z} = \begin{bmatrix} X \\ Y \end{bmatrix}$$

then the joint distribution is

$$f_{XY}(x, y) = f_{\mathbf{Z}}(\mathbf{z}) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\mathbf{z}^\top \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \mathbf{z}}$$

where $-1 < \rho = E[XY] < 1$

   (a) *(10 points)* What is the *Linear* MMSE estimate of Y given x?

   (b) *(10 points)* Please derive the maximum likelihood estimate of Y given x.

   (c) *(15 points)* Please derive the MMSE estimate of Y given x.

3. *(30 points)*

A discrete time linear system has a random input $u(t)$ with

$$p_{U(t)}(u(t)) = \begin{cases} \alpha & u(t) = 1 \\ 1-\alpha & u(t) = -1 \end{cases}$$

where $0 < \alpha < 1$. The $u(t)$ at different points in time are all mutually independent. The difference equation which describes the system is

$$x(t+1) = x(t) + u(t)$$

Assume $x(0) = 0$.

(a) *(10 points)* What is the probability that $x(n) = 0$, for n an integer greater than zero?

(b) *(10 points)* Please derive the Linear minimum mean square estimate for $x(T)$ based on the values of $x(t)$ for $t = 1, 2, ..., T-1$. Assume $x(0) = 0$ and $\alpha = 1/2$.

HINT: Don't just try to turn a crank.

(c) *(10 points)* Please derive the minimum mean square estimate for $x(T)$ based on the values of $x(t)$ for $t = 1, 2, ..., T-1$. Assume $x(0) = 0$ and $\alpha = 1/2$.

4. *(35 points)* In between solving the world's research problems, Rutgera Univera, the world famous Rutgers University graduate student shops for food at Infinite Food Emporium, a supermarket with infinite floorspace. The only thing finite about Infinite Food Emporium is the number of checkout lines, N. The lines are identical, idependent and can be as long as needed. The time a customer spends with a checkout clerk (cashier) is an exponential random variable with mean $1/\mu$.

(a) *(5 points)* Suppose people (including Rutgera) arrive to the checkout lines as a Poisson process with rate λ and then choose one of the N lines randomly. What value of service rate μ guarantees that the service system is stable (that queue lengths do not tend toward infinity)?

(b) *(5 points)* What is the steady state distribution of each line (number of customers in the line, including the one being served), assuming that the conditions established in the previous part on finite waiting time are satisfied?

(c) *(5 points)* Assume the store has been open a long time by the time Rutgera reaches the checkout line. What is Rutgera's mean waiting time (the time spent in the line before she begins service with the checkout clerk)?

(d) *(5 points)* Rutgera is the last customer to reach the checkout lanes. After she joins a queue, she notices that all the lines have exactly C customers. Assume there are no further arrivals to the checkout lanes. What is the probability that Rutgera's lane finishes first?

HINT: Remember that these are exponential servers.

(e) *(15 points)* Suppose Rutgera, instead of choosing a line at random, chooses the line with the least number of customers in it. What is her mean waiting time?