THE STATE UNIVERSITY OF NEW JERSEY

RUTGERS

College of Engineering
Department of Electrical and Computer Engineering

332:541 **Stochastic Signals and Systems** **Fall 2007**

Quiz II

There are 3 questions. You have at least the class period to answer them. Show all work. Answers given without work will receive no credit. GOOD LUCK!

1. *(30 points)* A random telegraph wave $X(t)$ flips back and forth between ± 1. The times between transitions are iid exponential random variables with parameter λ. We showed in class that $E[X(t)] = 0$ and $E[X(t)X(t+\tau)] = e^{-2\lambda|\tau|}$.

   $X(t)$ is applied to a linear time invariant filter with impulse response $h(t) = \delta(t) - \delta(t-1)$ where $\delta(t)$ is the unit impulse function. The output process is $Y(t)$.

   (a) *(10 points)* Please provide a simple analytic expression for $Y(t)$ in terms of $X(t)$.
   SOLUTION: $Y(t) = X(t) - X(t-1)$.

   (b) *(10 points)* Calculate $R_Y(\tau)$.
   SOLUTION:

$$\begin{aligned} R_Y(\tau) &= E[Y(t)Y(t+\tau)] \\ &= E[X(t)X(t+\tau)] + E[X(t-1)X(t-1+\tau)] \\ &- E[X(t)X[t-1+\tau] - E[X(t-1+\tau)X(t+\tau)] \end{aligned}$$

   Simplifying

$$R_Y(\tau) = 2R_X(\tau) - R_X(\tau-1) - R_X(\tau+1)$$

   (c) *(10 points)* The power spectral density of the output is $S_Y(f) = S_X(f)|H(f)|^2$. Calculate $S_X(f)$ and $H(f)$. Show that the resulting $S_Y(f)$ is equivalent to the Fourier transform of your result in the previous part.
   SOLUTION:

$$S_X(f) = \frac{1}{j2\pi f + 2\lambda} + \frac{1}{-j2\pi f + 2\lambda} = \frac{4\lambda}{(2\pi f)^2 + 4\lambda^2}$$

$$H(f) = \int_0^1 (\delta(t) - \delta(t-1))e^{-j2\pi ft}dt = 1 - e^{-j2\pi f} = 2je^{-j\pi f}\sin\pi f$$

   so that

$$|H(f)|^2 = 4\sin^2\pi f$$

   Thus

$$S_Y(f) = 4\sin^2\pi f\left(\frac{4\lambda}{(2\pi f)^2 + 4\lambda^2}\right)$$

The Fourier transform of $2R_X(\tau) - R_X(\tau-1) - R_X(\tau+1)$ *is*

$$S_X(f) = \left(2 - e^{-j2\pi f} - e^{j2\pi f}\right) S_X(f) = 2(1 - \cos 2\pi f) S_X(f) = 4\sin^2 \pi f S_X(f)$$

which is the desired result.

2. *(30 points)* We have learned that the best linear estimate of a random variable Y with mean μ_Y given an observation of a related random variable X with mean μ_X is

$$\hat{Y} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X^2}(X - \mu_X) + \mu_Y$$

(a) *(15 points)* Show that the expected square error $E[\mathcal{E}^2] = E[(\hat{Y} - Y)^2] = \sigma_Y^2(1 - \rho_{XY}^2)$ where $\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}$.

SOLUTION:

$$E[\mathcal{E}^2] = E[(\hat{Y} - Y)^2] = \left(\frac{Cov(X,Y)}{\sigma_X^2}\right)^2 \sigma_X^2 + \sigma_Y^2 - 2\frac{(Cov(X,Y))^2}{\sigma_X^2}$$

This reduces to

$$E[\mathcal{E}^2] = \sigma_Y^2 - \frac{(Cov(X,Y))^2}{\sigma_X^2} = \sigma_Y^2(1 - \rho_{XY}^2)$$

(b) *(15 points)* Show that if X and Y are jointly Gaussian random variables, the error $\mathcal{E} = \hat{Y} - Y$ is independent of the observation X. Is this independence a good or a bad thing in terms of the quality of the estimate?

SOLUTION: *$\mathcal{E}$ is a linear superposition of two jointly Gaussian random variables X and Y, so $\mathcal{E}$ is Gaussian and also jointly Gaussian with X.*

$$E[\mathcal{E}X] = E[\hat{Y}X - YX] = E[\frac{Cov(X,Y)}{\sigma_X^2}(X - \mu_X)X + \mu_Y X - YX]$$

so that

$$E[\mathcal{E}X] = \frac{Cov(X,Y)}{\sigma_X^2}(E[X^2] - \mu_X^2) + \mu_Y \mu_X - E[YX] = Cov(X,Y) - Cov(X,Y) = 0$$

Since $E[\mathcal{E}] = 0$, $\mathcal{E}$ and X are therefore independent.

This means that our estimator $\hat{Y}$ extracted all the possible information from the observable X so that what remains is independent of the observation – that is, the random error springs from uncertainty having nothing to do with the observed X. If, conversely, $\mathcal{E}$ and X were correlated, then X could still "tell us something about $\mathcal{E}$" and we could use that information to provide a better estimate of Y.

3. *(40 points)* Rutgera Univera, the world famous Rutgers University ECE graduate student has founded *CatBAM!* an early warning communications service which operates on the plains of Africa. Resourcefully, Rutgera uses available materials to build her network – specifically meercats, a type of African plains rodent and bat guano. Bat guano (droppings) are explosive.

Each meercat sentinel (watcher) is sent from his home base equipped with two guano-filled rockets. His job is to look for lions in the vicinity. If a lion is present, the meercat launches the first rocket. If no lion is present the meercat launches the second rocket. Both rockets will return to home base and explode to announce their arrival. If there is no wind, the first rocket will land at position $\mathbf{u}_{\text{lion}} = (1, -1)$. Likewise, the second rocket will land at $\mathbf{u}_{\overline{\text{lion}}} = (-1, 1)$.

However, there is ALWAYS gusty wind so that the actual landing position is a random vector $\mathbf{X} = \mathbf{u} + \mathbf{W}$ where $\mathbf{w}$ is a Gaussian random vector with mean zero and correlation matrix

$$\mathbf{R}_W = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$$

A decision about whether a lion is present or not is made from the observed landing position $\mathbf{X}$.

(a) *(10 points)* What are $f_{\mathbf{X}|\text{lion}}(\mathbf{x}|\text{lion})$, $f_{\mathbf{X}|\text{no-lion}}(\mathbf{x}|\text{no-lion})$?

SOLUTION: *The covariance matrix is diagonal so we have independent Gaussian random variables, offset by the mean landing position:*

$$f_{\mathbf{X}|lion}(\mathbf{x}|lion) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{(x_1-1)^2}{2\sigma_1^2}} e^{-\frac{(x_2+1)^2}{2\sigma_2^2}}$$

$$f_{\mathbf{X}|no\text{-}lion}(\mathbf{x}|\overline{no\text{-}lion}) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{(x_1+1)^2}{2\sigma_1^2}} e^{-\frac{(x_2-1)^2}{2\sigma_2^2}}$$

(b) *(10 points)* Assuming a lion is as likely as not to be present, please design a decision region on the observable $\mathbf{X}$ which minimizes the probability of error.

SOLUTION: *The likelihood ratio is*

$$\frac{\frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{(x_1-1)^2}{2\sigma_1^2}} e^{-\frac{(x_2+1)^2}{2\sigma_2^2}}}{\frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{(x_1+1)^2}{2\sigma_1^2}} e^{-\frac{(x_2-1)^2}{2\sigma_2^2}}} \begin{array}{c} Lion \\ > \\ < \\ No\ Lion \end{array} 1$$

which we simplify to

$$-\frac{(x_1-1)^2}{2\sigma_1^2} - \frac{(x_2+1)^2}{2\sigma_2^2} + \frac{(x_1+1)^2}{2\sigma_1^2} + \frac{(x_2-1)^2}{2\sigma_2^2} \begin{array}{c} Lion \\ > \\ < \\ No\ Lion \end{array} 0$$

which further simplifies to

$$\frac{x_1}{\sigma_1^2} \begin{array}{c} Lion \\ > \\ < \\ No\ Lion \end{array} \frac{x_2}{\sigma_2^2}$$

(c) *(5 points)* Carefully sketch the decision regions when $\sigma_1 = \sigma_2$. Carefully sketch the decision regions for $\sigma_1 = 2\sigma_2$.

SOLUTION: *With x_2 the ordinate (vertical axis) and x_1 the abscissa (horizontal axis), the region to the right of the line $x_1 = x_2$ is the "say Lion" region when $\sigma_1 = \sigma_2$. The "say Lion" region is to the right of the line $x_1 = 4x_2$ when $\sigma_1 = 2\sigma_2$.*

(d) *(15 points)* Now suppose the effects of the gusts are correlated with the rocket. That is, suppose that

$$\mathbf{R}_{W|\text{lion}} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_1^2 \end{bmatrix}$$

and

$$\mathbf{R}_{W|\text{no lion}} = \begin{bmatrix} \sigma_2^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$$

Find and sketch the decision regions when $\sigma_2^2 = 1$ and $\sigma_1^2 = 5/3$.

HINT: Wait until you've simplified your analytic description of the decision region before doing the substitution for the variances.

SOLUTION: *The likelihood ratio is*

$$\frac{\frac{1}{2\pi\sigma_1^2} e^{-\frac{(x_1-1)^2}{2\sigma_1^2}} e^{-\frac{(x_2+1)^2}{2\sigma_1^2}}}{\frac{1}{2\pi\sigma_2^2} e^{-\frac{(x_1+1)^2}{2\sigma_2^2}} e^{-\frac{(x_2-1)^2}{2\sigma_2^2}}} \underset{\text{No Lion}}{\overset{\text{Lion}}{\gtrless}} 1$$

so that the log likelihood ratio is

$$-\frac{(x_1-1)^2}{2\sigma_1^2} - \frac{(x_2+1)^2}{2\sigma_1^2} + \frac{(x_1+1)^2}{2\sigma_2^2} + \frac{(x_2-1)^2}{2\sigma_2^2} \underset{\text{No Lion}}{\overset{\text{Lion}}{\gtrless}} 2\log\frac{\sigma_1}{\sigma_2}$$

Rearranging

$$-\sigma_2^2(x_1-1)^2 + \sigma_1^2(x_1+1)^2 - \sigma_2^2(x_2+1)^2 + \sigma_1^2(x_2-1)^2 \underset{\text{No Lion}}{\overset{\text{Lion}}{\gtrless}} 4\sigma_1^2\sigma_2^2\log\frac{\sigma_1}{\sigma_2}$$

and combining

$$(x_1^2 + 2x_1\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2-\sigma_2^2} + 1) + (x_2^2 - 2x_2\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2-\sigma_2^2} + 1) \underset{\text{No Lion}}{\overset{\text{Lion}}{\gtrless}} 4\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2-\sigma_2^2}\log\frac{\sigma_1}{\sigma_2}$$

Completing the squares we have

$$\left(x_1 + \frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2-\sigma_2^2}\right)^2 + \left(x_2 - \frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2-\sigma_2^2}\right)^2 + 2\left(1 - \left(\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2-\sigma_2^2}\right)^2\right) \underset{\text{No Lion}}{\overset{\text{Lion}}{\gtrless}} 4\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2-\sigma_2^2}\log\frac{\sigma_1}{\sigma_2}$$

Doing the substitutions yields:

$$(x_1+4)^2 + (x_2-4)^2 \underset{\text{No Lion}}{\overset{\text{Lion}}{\gtrless}} 10\log\frac{5}{3} + 30$$

Thus, No-Lion should be the decision if $\mathbf{x}$ *is within a circle centered at* $(-4,4)$ *of radius* $\sqrt{10\log\frac{5}{3}+30}$.