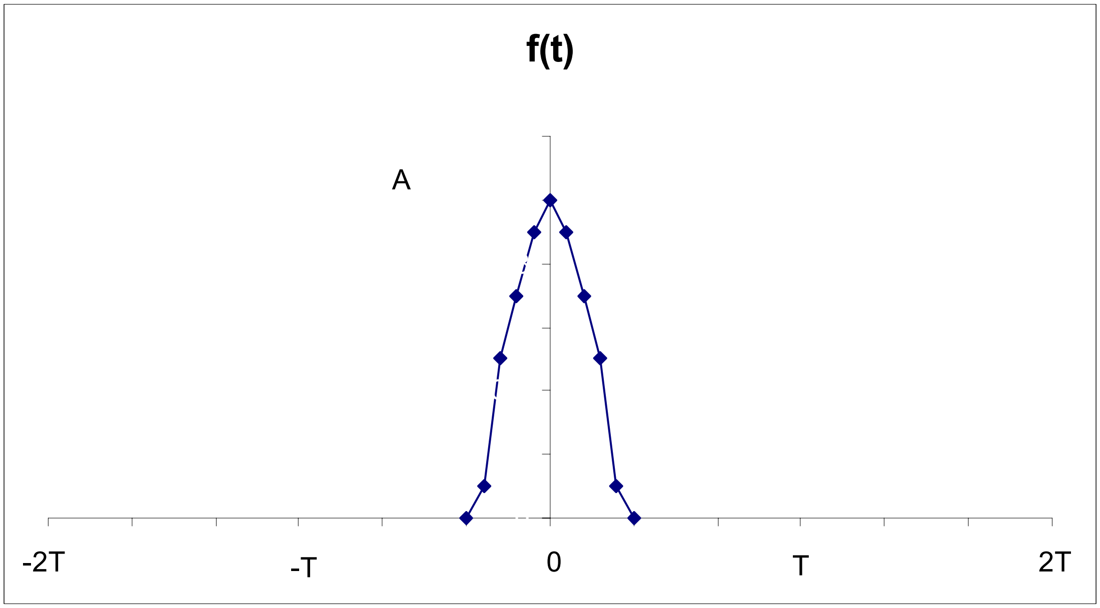
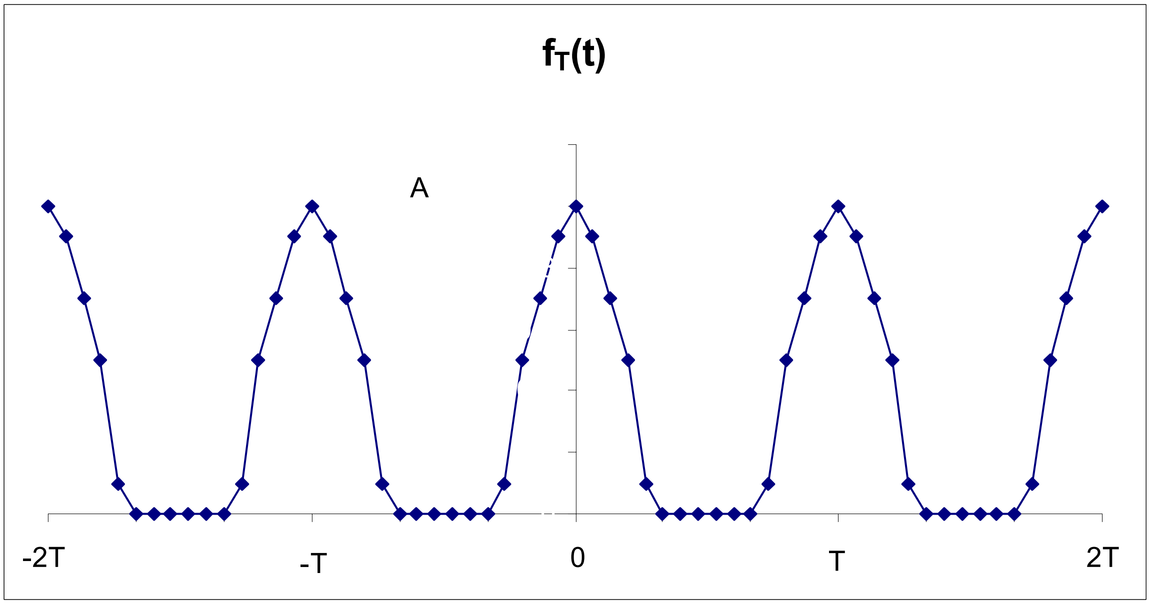
The Fourier Transform

Derivation

Assume that we have a generalized, time-limited pulse centered at t = 0 as shown below.



The Fourier Transform of this pulse can be developed by starting with a periodic version of this pulse where the original pulse now repeats every T seconds.



Note:

$$\lim_{T \longrightarrow \infty} f_T(t) = f(t)$$

$f_T(t)$ is periodic with period T so we can express it by its exponential Fourier series as

$$f_T(t) = \sum_{n=-\infty}^{\infty} F_n * \varepsilon^{jn\omega_0 t}$$

where

$$F_n = \frac{1}{T} \int_{-T/2}^{T/2} f_T(t) * \varepsilon^{-jn\omega_0 t} dt$$

and

$$\omega_0 = 2\pi/T$$

Now let's make a small change in notation

1. $\omega_n = n*\omega_0$
2. $F(\omega_n) = T*F_n$

We now have

$$f_T(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} F(\omega_n) * \varepsilon^{j\omega_n t} \quad \text{and} \quad F_n = \int_{-T/2}^{T/2} f_T(t) * \varepsilon^{-j\omega_n t} dt$$

The sum can be rewritten as

$$f_T(t) = \frac{\omega_0}{2\pi} \sum_{n=-\infty}^{\infty} F(\omega_n) * \varepsilon^{j\omega_n t}$$

or

$$f_T(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F(\omega_n) * \varepsilon^{j\omega_n t} \omega_0$$

Taking the limit as $T \longrightarrow \infty$

$$\lim_{T \longrightarrow \infty} f_T(t) = f(t) = \frac{1}{2\pi} \lim_{T \longrightarrow \infty} \left[\sum_{n=-\infty}^{\infty} F(\omega_n) * \varepsilon^{j\omega_n t} \omega_0 \right]$$

But $\omega_0 = 2\pi/T$ so for large T let $\omega_0 \longrightarrow \Delta\omega$ and the limit becomes

$$f(t) = \frac{1}{2\pi} \lim_{T \longrightarrow \infty} \left[\sum_{n=-\infty}^{\infty} F(\omega_n) * \varepsilon^{j\omega_n t} \Delta\omega \right]$$

or since $T \longrightarrow \infty$ implies that $\Delta\omega \longrightarrow 0$ and the sum, in the limit, becomes an integral

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega\,) * \varepsilon^{\,j\omega\,t}\,d\omega \quad \text{and} \qquad F\,(\omega) = \int_{-\infty}^{\infty} f_T(t) * \varepsilon^{-j\omega\,t}\,dt$$

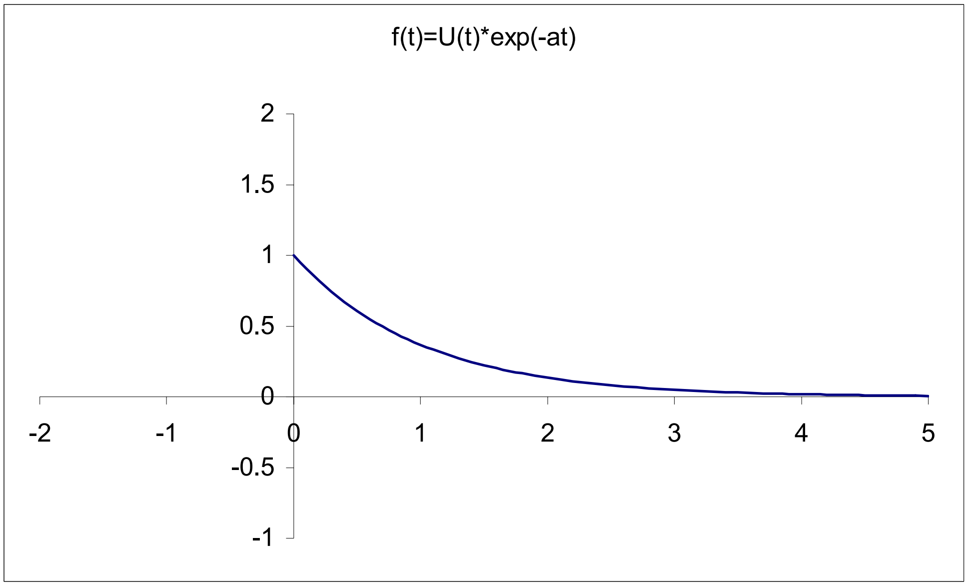
This pair of equations defines the Fourier Transform

1. F(ω) is the **Fourier Transform** of f(t)
2. f(t) is the inverse **Fourier Transform** of F(ω)
3. F(ω) is also called the **Spectral Density** of f(t) as it describes how the energy of the original pulse is distributed as a function of frequency (in radians per second)

I use a backwards upper case script “F” to denote taking the Fourier Transform of a function and the same symbol with a “-1” superscript to denote taking the inverse Fourier Transform.

Example 1

Take the Fourier Transform of the single-sided exponential



$$F(\omega) = \int_{-\infty}^{\infty} U(t) * \varepsilon^{-at} \varepsilon^{j\omega t}\, dt$$

$$F(\omega) = \int_{0}^{\infty} \varepsilon^{-at} \varepsilon^{-j\omega t}\, dt$$

$$F(\omega) = \int_{0}^{\infty} \varepsilon^{-(a+j\omega)t}\, dt$$

$$F(\omega) = \frac{-1}{a+j\omega} * \varepsilon^{-(a+j\omega)t} \Big|_{0}^{\infty}$$

$$F(\omega) = \frac{1}{a+j\omega}$$

Note that the Fourier Transform is complex. It has a magnitude and a phase. The magnitude is found by multiplying it by its complex conjugate and taking the square root.

$$\left|F(\omega)\right|^2 = \frac{1}{a+j\omega} * \frac{1}{a-j\omega}$$

$$\left|F(\omega)\right|^2 = \frac{1}{a^2+\omega^2}$$

$$\left|F(\omega)\right| = \frac{1}{\sqrt{a^2+\omega^2}}$$ This is the magnitude

Now find the phase. First, find the real and imaginary parts.

$$F(\omega)=\frac{1}{a+j\omega}$$

$$F(\omega)=\frac{1}{a+j\omega}*\frac{a-j\omega}{a-j\omega}$$

$$F(\omega)=\frac{a-j\omega}{a^2+\omega^2}=\frac{a}{a^2+\omega^2}-\frac{j\omega}{a^2+\omega^2}$$

Therefore the real part is

$$\mathrm{Re}\left[F(\omega)\right]=\frac{a}{a^2+\omega^2}$$

and the imaginary part is

$$\mathrm{Im}\left[F(\omega)\right]=\frac{-\omega}{a^2+\omega^2}$$

The phase is then given by

$$\theta=\tan^{-1}\left[\frac{\mathrm{Im}\left[F(\omega)\right]}{\mathrm{Re}\left[F(\omega)\right]}\right]=-\tan^{-1}\left[\frac{\omega}{a}\right]$$

Singularity Functions

We run into special functions when taking the Fourier Transform of functions that have infinite energy. The first of these special functions is the **Delta Function**

$$\partial(t) = \lim_{\varepsilon \longrightarrow \infty} G_\varepsilon(t)$$

Where $G_\varepsilon(t)$ is any function from the set of all functions having the properties

1. $\int_{-\infty}^{\infty} G_\varepsilon(t)dt = 1$
2. $\lim_{\varepsilon \longrightarrow \infty} G_\varepsilon(t) = 0$ For all $t \neq 0$

Sifting Property of the Delta Function

Integrating the product of the Delta Function with a "well-behaved" function results in "sampling" the "well-behaved" function at the time that the Delta Function goes to infinity. Or

$$\int_a^b f(t) * \partial(t - t_0)dt = \langle \begin{matrix} f(t_0) & \text{if} & a<t_0<b \\ 0 & & \text{eleswhere} \end{matrix}$$

Proof

Use Integration by parts

$$\int_a^b U(t)dV(t) = U(t)V(t)\Big|_a^b - \int_a^b V(t)dU(t)$$

Let $U(t) = f(t)$ and $dV(t) = \delta(t-t_0)dt$

$$\int_a^b f(t) * \partial(t - t_0)dt = f(t)U(t - t_0)\Big|_a^b - \int_a^b f^{'}(t) * U(t)dt$$

Case 1: $a < t_0 < b$

$$\int_a^b f(t) * \partial(t - t_0)dt = f(b) - 0 - \int_{t_0}^b f^{'}(t) * U(t)dt$$

$$\int_a^b f(t) * \partial(t - t_0)dt = f(b) - f(t)\Big|_{t_0}^b$$

$$\int_a^b f(t) * \partial(t - t_0)dt = f(b) - f(b) + f(t_0)$$

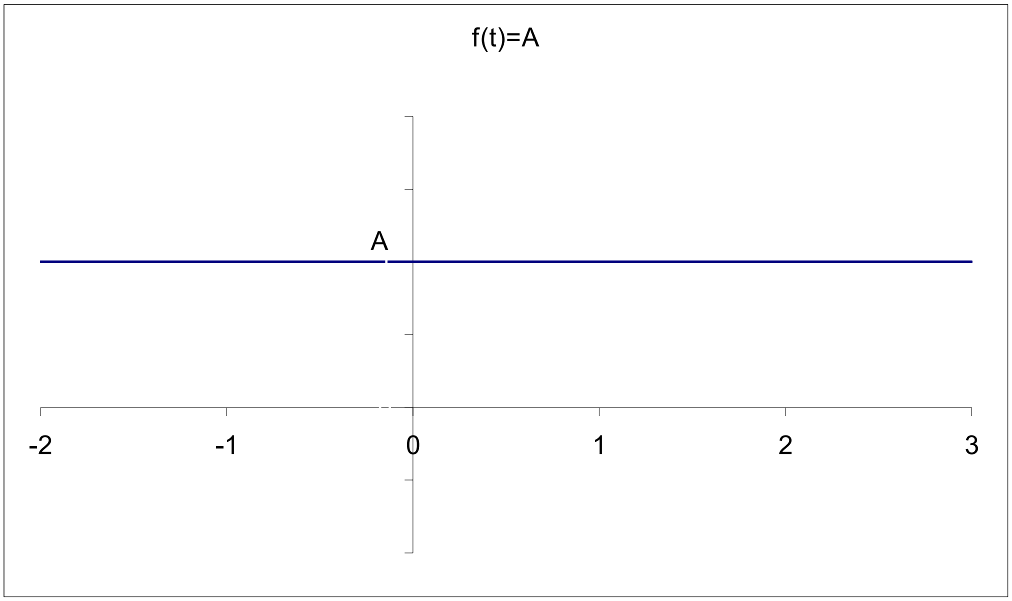
$$\int_a^b f(t) * \partial(t - t_0)dt = f(t_0) \qquad \text{Q.E.D}$$

Case 2: $t_0 < a$ or $t_0 > b$

$$\int_a^b f(t) * \partial(t - t_0)dt = 0 - 0 - \int_a^b 0dt = 0 \text{ Q.E.D}$$

Example 2

Take the Fourier Transform of a constant



$$F(\omega) = \int_{-\infty}^{\infty} A\varepsilon^{j\omega t}\,dt$$

Here the integral can't be directly computed, we have to approach it as a limiting case. Let's replace the constant with a parameterized function that equals the constant as its parameter approaches zero, the double-sided exponential function:

$$f(t) = A\varepsilon^{-a|t|}$$

Now the Transform becomes:

$$F_a(\omega) = \int_{-\infty}^{\infty} A\varepsilon^{-a|t|}\varepsilon^{j\omega t}\,dt = \int_{-\infty}^{0} A\varepsilon^{-at}\varepsilon^{j\omega t}\,dt + \int_{0}^{\infty} A\varepsilon^{-at}\varepsilon^{j\omega t}\,dt$$

Let u = -ω in the first integral

$$F_a(\omega) = \int_{\infty}^{0} A\varepsilon^{-at}\varepsilon^{j(-u)t}\,dt + \int_{0}^{\infty} A\varepsilon^{-at}\varepsilon^{j\omega t}\,dt$$

From our first example this is:

$$F_a(\omega) = \frac{A}{a - j\omega} + \frac{A}{a + j\omega} = \frac{2Aa}{a^2 + \omega^2}$$

Now we need to take the limit as a ⟶ 0 to get F(ω)

$$F(\omega) = \lim_{a \longrightarrow 0} F_a(\omega)$$

$$F(\omega) = \lim_{a \longrightarrow 0} \frac{2Aa}{a^2 + \omega^2} = \left\langle \begin{array}{ll} 0 & if\ \omega \neq 0 \\ \\ \infty & if\ \omega = 0 \end{array} \right.$$

so this is a δ-function that goes to ∞ at ω = 0 if its integral is a constant.

$$I = \int_{-\infty}^{\infty} 2A \frac{a}{a^2 + \omega^2} d\omega$$

Let a*x = ω

$$I = 2A \int_{-\infty}^{\infty} \frac{a}{a^2 \left(1 + x^2\right)} a dx$$

$$I = 2A \int_{-\infty}^{\infty} \frac{1}{1 + x^2} dx$$

$$I = 2A * \tan^{-1} x \, \Big|_{-\infty}^{\infty}$$

$$I = 2A * \left[\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right]$$

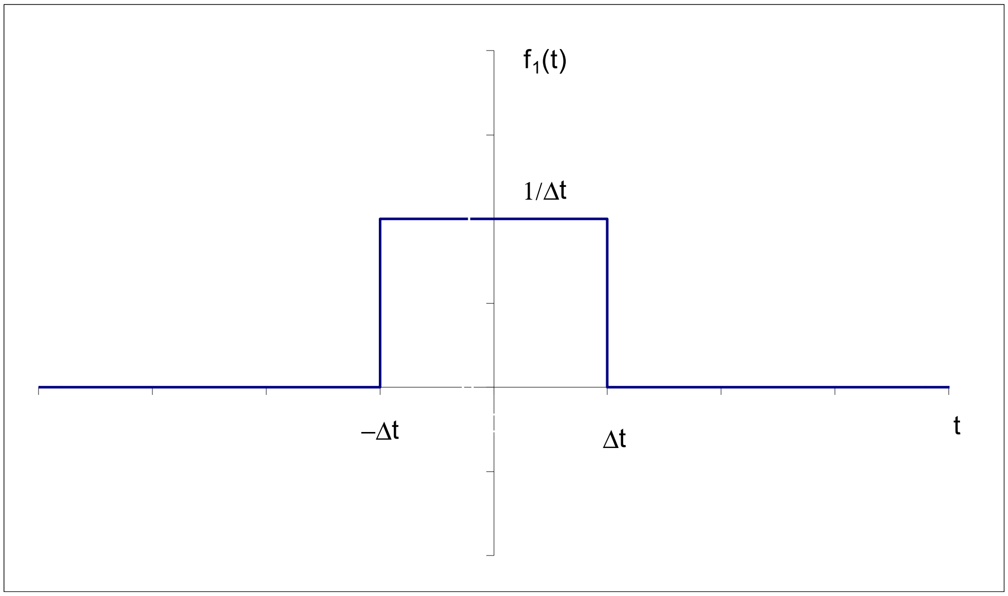
$$I = 2\pi A$$

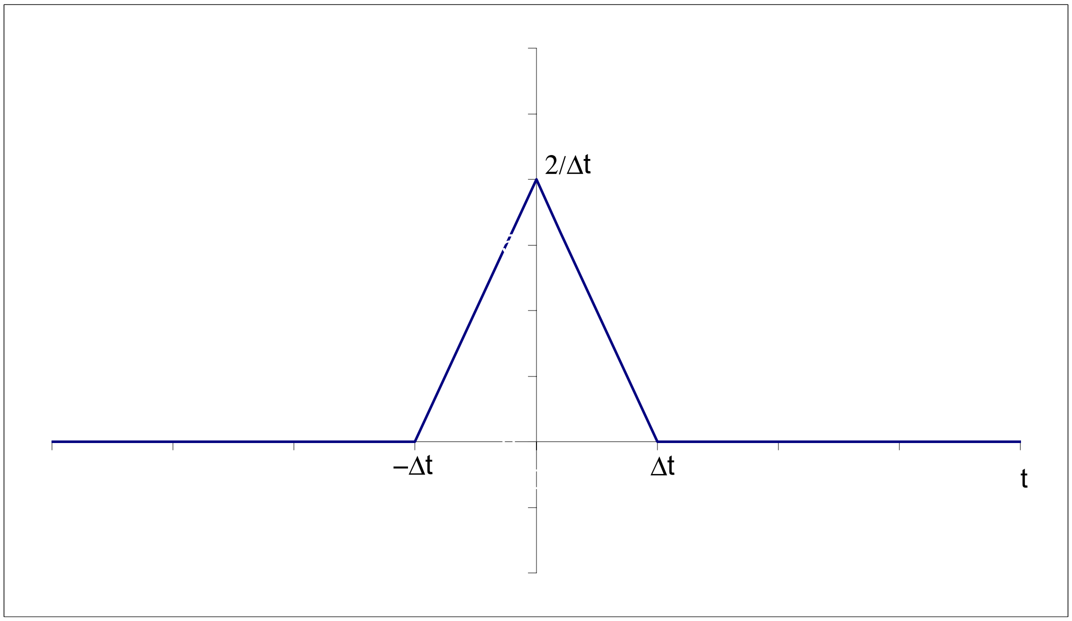
Therefore

$$\boxed{F(\omega) = 2\pi A * \partial(\omega)}$$

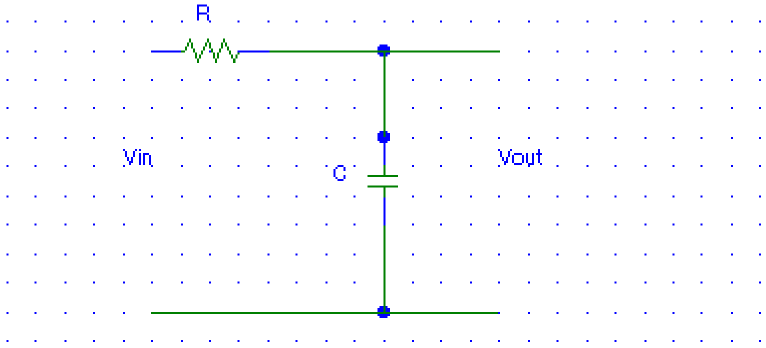
Exercises:

1: Find the Fourier Transforms for each of the two pulses





2: Find the transfer function for the simple RC low-pass filter



3: Determine the Fourier Transform of the RC low-pass filter output due to each of the pulses in part 1

4: Find the limit of each of the results in part 3 as $\Delta t \longrightarrow 0$

Properties of the Fourier Transform

Symmetry Property

If $f(t) \longleftrightarrow F(\omega)$

Then $F(t) \longleftrightarrow 2\pi\, f(-\omega)$

Proof:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)*\varepsilon^{j\omega t}\,d\omega$$

Therefore

$$2\pi * f(-t) = \int_{-\infty}^{\infty} F(\omega)\varepsilon^{-j\omega t}\,d\omega$$

Let u = ω and v = t

$$2\pi * f(-v) = \int_{-\infty}^{\infty} F(u)\varepsilon^{-juv}\,du$$

Now let ω = v and t = u

$$2\pi * f(-\omega) = \int_{-\infty}^{\infty} F(t)\varepsilon^{-j\omega t}\,dt$$

Therefore $F(t) \longleftrightarrow 2\pi\, f(-\omega)$

And if f(t) is an even function

$F(t) \longleftrightarrow 2\pi\, f(\omega)$

Linearity Property

If $f_1(t) \longleftrightarrow F_1(\omega)$

And $f_2(t) \longleftrightarrow F_2(\omega)$

Then $[a*f_1(t) + b*f_1(t)] \longleftrightarrow [a*F_1(\omega) + b*F_2(\omega)]$

Proof:

Results due to the linearity of integration

Scaling Property

If f(t) $\longleftrightarrow$ F(ω)

Then for a real

$$f(a*t) \longleftrightarrow \frac{1}{|a|}F\left(\frac{\omega}{a}\right)$$

Proof:

$$\Im\{f(a*t)\} = \int_{-\infty}^{\infty} f(a*t)\varepsilon^{-j\omega t}\,dt$$

case 1: a > 0 Let x = a*t

$$\Im\{f(a*t)\} = \int_{-\infty}^{\infty} f(x)\varepsilon^{-j\frac{\omega}{a}x}\frac{1}{a}\,dx$$

$$\Im\{f(a*t)\} = \frac{1}{a}\int_{-\infty}^{\infty} f(x)\varepsilon^{-j\frac{\omega}{a}x}\,dx$$

or

$$\Im\{f(a*t)\} = \frac{1}{a}F\left(\frac{\omega}{a}\right)$$

case 2: a < 0 Again let x = a*t

$$\Im\{f(a*t)\} = \int_{\infty}^{-\infty} f(x)\varepsilon^{-j\frac{\omega}{a}x}\frac{1}{a}\,dx$$ (Note the limits are now backwards)

$$\Im\{f(a*t)\} = -\frac{1}{a}\int_{-\infty}^{\infty} f(x)\varepsilon^{-j\frac{\omega}{a}x}\,dx$$

or

$$\Im\{f(a*t)\} = -\frac{1}{a}F\left(\frac{\omega}{a}\right)$$

Therefore including both cases

$$f(a*t) \longleftrightarrow \frac{1}{|a|}F\left(\frac{\omega}{a}\right)$$

Q. E. D.

Note: The compression of a function in the time domain results in an expansion in the frequency domain and vice versa.

Frequency Shifting

If $f(t) \leftrightarrow F(\omega)$

Then $f(t) * \varepsilon^{j\omega_0 t} \leftrightarrow F(\omega - \omega_0)$

Proof:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) * \varepsilon^{-j\omega t}\, dt$$

$$F(\omega - \omega_0) = \int_{-\infty}^{\infty} f(t) * \varepsilon^{-j(\omega-\omega_0)t}\, dt$$

$$F(\omega - \omega_0) = \int_{-\infty}^{\infty} \left[f(t) * \varepsilon^{j\omega_0 t} \right] * \varepsilon^{-j\omega t}\, dt$$

or

$$F(\omega - \omega_0) = \Im\left\{ f(t) * \varepsilon^{j\omega_0 t} \right\}$$

Q. E. D.

Note: The Modulation Theorem (very important in communications)

Remember Euler's Identities

$$\cos(x) = \frac{\varepsilon^{jx} + \varepsilon^{-jx}}{2} \quad \text{and} \quad \sin(x) = \frac{\varepsilon^{jx} - \varepsilon^{-jx}}{2j}$$

therefore

$$f(t)\cos(x) = \frac{f(t) * \varepsilon^{jx} + f(t) * \varepsilon^{-jx}}{2}$$

or

$$\boxed{f(t)\cos(x) \longleftrightarrow \frac{F(\omega + \omega_0) + F(\omega - \omega_0)}{2}}$$

similarly

$$f(t)\sin(x) = \frac{f(t) * \varepsilon^{jx} - f(t) * \varepsilon^{-jx}}{2j}$$

or

$$\boxed{f(t)\sin(x) \longleftrightarrow j\,\frac{F(\omega + \omega_0) - F(\omega - \omega_0)}{2}}$$

Time Shifting

If $f(t) \leftrightarrow F(\omega)$

Then $f(t-t_0) \leftrightarrow F(\omega_0) * \varepsilon^{-j\omega t_0}$

Proof:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) * \varepsilon^{j\omega t} d\omega$$

$$f(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) * \varepsilon^{j\omega(t-t_0)} d\omega$$

$$f(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[F(\omega) * \varepsilon^{-j\omega t_0}\right] * \varepsilon^{j\omega t} d\omega$$

$$f(t-t_0) \leftrightarrow F(\omega) * \varepsilon^{-j\omega t_0}$$

Q. E. D.

Time Differentiation and Integration

If $f(t) \leftrightarrow F(\omega)$

Then $\frac{d}{dt}[f(t)] \leftrightarrow (j\omega)F(\omega)$

And $\int_{-\infty}^{t} f(\tau)d\tau \leftrightarrow \frac{1}{j\omega}F(\omega)$

Proof:

First for differentiation (part 1)

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) * \varepsilon^{j\omega t} d\omega$$

$$\frac{d}{dt}[f(t)] = \frac{d}{dt}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) * \varepsilon^{j\omega t} d\omega\right]$$

$$\frac{d}{dt}[f(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) * \frac{d}{dt}\left[\varepsilon^{j\omega t}\right] d\omega$$

$$\frac{d}{dt}[f(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) * j\omega * \varepsilon^{j\omega t} d\omega$$

$$\frac{d}{dt}[f(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} [(j\omega)F(\omega)] * \varepsilon^{j\omega t} d\omega$$

or

$\frac{d}{dt}[f(t)] \leftrightarrow (j\omega)F(\omega)$ Q. E. D. for part 1

Now for integration (part 2)

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)*\varepsilon^{j\omega t}d\omega$$

$$\int_{-\infty}^{t}f(\tau)d\tau=\int_{-\infty}^{t}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)*\varepsilon^{j\omega\tau}d\omega\right]d\tau$$

interchanging the order of integration

$$\int_{-\infty}^{t}f(\tau)d\tau=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)*\left[\int_{-\infty}^{t}\varepsilon^{j\omega\tau}d\tau\right]d\omega$$

$$\int_{-\infty}^{t}f(\tau)d\tau=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)*\left[\frac{1}{j\omega}\varepsilon^{j\omega t}\right]d\omega$$

$$\int_{-\infty}^{t}f(\tau)d\tau=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{j\omega}F(\omega)\right]*\varepsilon^{j\omega t}d\omega$$

or

$$\int_{-\infty}^{t}f(\tau)d\tau\leftrightarrow\frac{1}{j\omega}F(\omega)$$ Q. E. D. for part 2

Frequency Differentiation

If $f(t) \leftrightarrow F(\omega)$

Then $(-jt)^n f(t) \leftrightarrow \dfrac{d^n}{dt^n} F(\omega)$

Proof:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) * \varepsilon^{-j\omega t} dt$$

$$\frac{d^n}{dt^n} F(\omega) = \frac{d^n}{dt^n}\left[\int_{-\infty}^{\infty} f(t) * \varepsilon^{-j\omega t} dt\right]$$

$$\frac{d^n}{dt^n} F(\omega) = \int_{-\infty}^{\infty} f(t) * \frac{d^n}{dt^n}\left[\varepsilon^{-j\omega t}\right] dt$$

$$\frac{d^n}{dt^n} F(\omega) = \int_{-\infty}^{\infty} f(t) * (-jt)^n \varepsilon^{-j\omega t} dt$$

$$\frac{d^n}{dt^n} F(\omega) = \int_{-\infty}^{\infty} \left[(-jt)^n * f(t)\right] * \varepsilon^{-j\omega t} dt$$

or

$$(-jt)^n f(t) \leftrightarrow \frac{d^n}{dt^n} F(\omega) \qquad \text{Q. E. D.}$$

The Convolution Theorem

Definition: the convolution of two functions $f_1(t)$ *and* $f_2(t)$ is defined as:

$$f_1(t) \otimes f_2(t) \equiv \int_{-\infty}^{\infty} f_1(\tau) * f_2(t-\tau) d\tau = \int_{-\infty}^{\infty} f_2(\tau) * f_1(t-\tau) d\tau$$

Time Convolution

If $f_1(t) \leftrightarrow F_1(\omega)$

And $f_2(t) \leftrightarrow F_2(\omega)$

Then $\Im\{f_1(t) \otimes f_2(t)\} \leftrightarrow F_1(\omega) * F_2(\omega)$

Proof:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) * \varepsilon^{-j\omega t} dt$$

Therefore

$$\Im\{f_1(t) \otimes f_2(t)\} = \int_{t=-\infty}^{\infty} \varepsilon^{-j\omega t} \left[\int_{\tau=-\infty}^{\infty} f_1(\tau) * f_2(t-\tau) d\tau \right] dt$$

$$\Im\{f_1(t) \otimes f_2(t)\} = \int_{\tau=-\infty}^{\infty} f_1(\tau) \left[\int_{t=-\infty}^{\infty} f_2(t-\tau) * \varepsilon^{-j\omega t} dt \right] d\tau$$

Let u = t - τ in the inner integral

$$\Im\{f_1(t) \otimes f_2(t)\} = \int_{t=-\infty}^{\infty} f_1(\tau) \left[\int_{u=-\infty}^{\infty} f_2(u) * \varepsilon^{-j\omega(u+\tau)} du \right] d\tau$$

$$\Im\{f_1(t) \otimes f_2(t)\} = \int_{t=-\infty}^{\infty} f_1(\tau) \varepsilon^{-j\omega\tau} \left[\int_{u=-\infty}^{\infty} f_2(u) * \varepsilon^{-j\omega u} du \right] d\tau$$

Since the inner integral is no longer a function of τ, it can be brought out as a constant and this leaves

$$\Im\{f_1(t) \otimes f_2(t)\} = \int_{t=-\infty}^{\infty} f_1(\tau) \varepsilon^{-j\omega\tau} d\tau * \int_{u=-\infty}^{\infty} f_2(u) * \varepsilon^{-j\omega u} du$$

or

$\Im\{f_1(t) \otimes f_2(t)\} = \Im\{f_1(\tau)\} * \Im\{f_2(u)\}$ Q. E. D

Frequency Convolution

If $f_1(t) \leftrightarrow F_1(\omega)$

And $f_2(t) \leftrightarrow F_2(\omega)$

Then $f_1(t) * f_2(t) \leftrightarrow \frac{1}{2\pi} F_1(\omega) \otimes F_2(\omega)$

Proof: Same method as for time convolution